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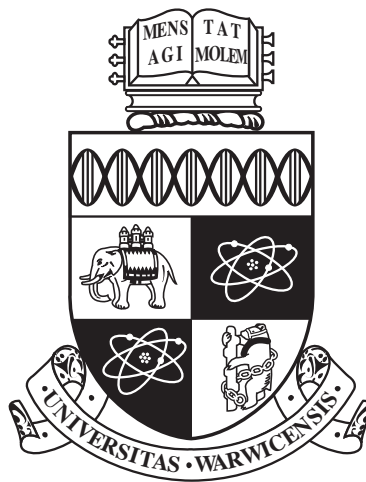
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**Primitive stability and Bowditch conditions for  
rank 2 free group representations**

by

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**Thesis**

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# Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. The work presented (including data generated and data analysis) was carried out by the author, except in the case of Figure 5.1, which has been originally produced by C. Series, S. P. Tan and Y. Yamashita, and published in [STY].

# Abstract

We introduce two different properties of representations of the free group of rank 2 into the group of hyperbolic isometries  $\mathbb{P}SL(2, \mathbb{C})$ : BQ conditions and primitive stability. We investigate relations between the sets of characters satisfying these conditions, and study certain slices of the character variety separately. The results we get are in line with the conjecture that these two sets actually coincide. We also show that in the diagonal slice of the character variety, namely the set of representations with associated trace triple  $(z, z, z)$ ,  $z \in \mathbb{C}$ , there exists a large class of primitive stable representations which are not discrete and faithful.



# Chapter 1

## Introduction

Hyperbolic manifolds have been a big source of interest and study in the last decades. The idea of a hyperbolic geometry was born in the first half of the nineteenth century. From the very first definitions in the context of complex analysis, this rich branch of mathematics gradually developed mainly in three directions, trying to understand and explain various phenomena that can be partitioned into three big classes: the theory of complex variables and conformal mappings, the topology of hyperbolic manifolds and the combinatorics of groups. The three brilliant minds contributing to these different yet intertwined fields belong to Henri Poincaré, and his original ideas from where most of the theory originated, William Thurston, with his remarkable geometrisation conjecture, and Mikhail Gromov, with his coarse point of view on geometric group theory.

In this thesis, we shall mainly focus on the topology of hyperbolic surfaces and 3-manifolds and, to a smaller extent, on some group theoretical features of the free groups. A hyperbolic orbifold can be described as the quotient of the hyperbolic space under the action of a discrete group of isometries. Such a group is called a *Kleinian group*. If the group is also torsion-free, then the quotient is a hyperbolic manifold. For this reason, the geometry of hyperbolic orbifolds and the algebra of Kleinian groups are deeply related.

Hyperbolic isometries are bijectively associated to elements in the group  $\mathbb{P}SL(2, \mathbb{C})$ , and therefore they can be thought of as (classes of) matrices. When we want to analyse many isomorphic Kleinian groups (isomorphic in the classical sense, that is in the category of groups), a very powerful tool is to think of them as images of the abstract group to which they are isomorphic. That is, we fix a group and consider its *representations* into  $\mathbb{P}SL(2, \mathbb{C})$ . In other words, we are describing a concrete subgroup of  $\mathbb{P}SL(2, \mathbb{C})$  as the image of a homomorphism from

an “external” group. This is the most natural setting to allow *deformations* of Kleinian groups. Deforming a Kleinian group roughly means continuously varying the image of a fixed abstract group into the space of hyperbolic isometries. Now that a Kleinian group has been replaced by a *map*, the next natural step to take is to consider the space of all of these maps. For an abstract group  $G$ , we form the space  $Hom(G, \mathbb{P}SL(2, \mathbb{C})) // \mathbb{P}SL(2, \mathbb{C})$  of (roughly) conjugacy classes of homomorphisms from  $G$  to the group of hyperbolic isometries, and call it *character variety*. We shall be interested in the very special case where  $G$  is a free group of rank 2.

We shall be interested in subsets of the character variety satisfying certain specific conditions.

The first subset we shall encounter is that of the representations satisfying BQ conditions, or Bowditch conditions (see Definition 2.4 for a definition). We shall denote this open subset by the symbol  $\mathcal{BQ}$ . BQ conditions were introduced by Bowditch in [Bow]. In this paper, the author analyses connections between Markoff triples and representations of the free group of rank 2, and proves that in the subset of characters satisfying BQ conditions, some identity “à la McShane” holds. The same subspace is deeply studied in [TWZ], where the authors obtain a further extension of McShane’s identity and prove that the mapping class group of the torus acts properly discontinuously on it. BQ conditions can be described purely in terms of traces of matrices, and can be therefore considered algebraic in flavour. Nevertheless, the trace of a loxodromic element matrix has an important interpretation in terms of its translation length, and we shall see the geometrical repercussions of this key fact.

On the geometrical side of the game, instead, lies a different property of characters, called primitive stability. This concept was introduced by Minsky in [Min] and it is basically has to do with hyperbolic geometry, being defined in terms of Cayley graphs, metrics and quasi geodesics (see Definition 2.7). In [Min], the author is interested in studying the dynamics of the action of  $Out(F_n)$ , the outer automorphism group of the free group of rank  $n$ , on the character variety  $Hom(F_n, \mathbb{P}SL(2, \mathbb{C})) // \mathbb{P}SL(2, \mathbb{C})$ . The main theorem of the paper states the existence of an open subset of the character variety, strictly larger than the set of Schottky representations, where  $Out(F_n)$  acts properly discontinuously. This is exactly the subset of primitive stable characters, that we shall denote by  $\mathcal{PS}$ .

Rather oddly (or maybe not!) when  $n = 2$ ,  $Out(F_2)$  is isomorphic to the mapping class group of the torus. Moreover, both  $\mathcal{BQ}$  and  $\mathcal{PS}$  are open sets and contain all the Schottky characters. Assuming that there may be a correlation between these apparently different sets is quite natural. This leads to one of the two

main conjectures in this topic, that is  $\mathcal{PS} = \mathcal{BQ}$ . This conjecture doesn't seem to be proved so far. Another important open conjecture, made by Bowditch, states that, in the case of representations of  $F_2$  where the commutator element is sent to a parabolic, BQ conditions correspond to quasi Fuchsian representations. Although both conjectures seem to be still open at the time of writing, we shall give an argument to see that primitive stability implies BQ conditions, and shall prove that  $\mathcal{PS}$  also contains a class of representations which is larger than Schottky characters, reinforcing Minsky's observation. These preliminary results will be exposed in Chapter 2.

As often occurs in mathematics, when one has to tackle a problem in a space that is too big and complicated to be analysed at once, a clever way of proceeding is to restrict to a few subcases and try to solve the problem within these easier settings first, hoping that they might be correlated and eventually they could reveal the big picture. Investigating BQ conditions and primitive stability in the whole character variety is a hard task. In order to work in a smaller, more easily understandable setting, we will play this trick and focus our efforts on some specific subsets.

In Chapter 3, we only consider representations whose image is contained in  $PSL(2, \mathbb{R})$ . This allows us to work with two dimensional hyperbolic geometry, and to develop some technique to prove primitive stability only by looking at the trace of the commutator element.

If  $a$  and  $b$  form a pair of generators of the free group of rank 2, the conjugacy class of a character is uniquely determined by the three complex values of the traces of the image matrices corresponding to  $a$ ,  $b$  and  $ab$ . This allows us to identify the character variety with the complex vector space  $\mathbb{C}^3$ . The identification with  $\mathbb{C}^3$  helps us in easily slicing the variety again. If the triple of traces is real, for instance, we say that the associated character is a *real representation*. Thus, the subspace of real representations can be seen as the real three-dimensional vector space sitting inside the complex one,  $\mathbb{R}^3 \subseteq \mathbb{C}^3$ . We shall analyse primitive stability and BQ conditions in this space in Chapter 4, following the outline that Goldman gives in [Gol2], and using many of his results. In [Gol2], the author carefully analyses real representations by looking at the trace of the commutator element, and studies how the action of the mapping class group changes according to this trace. Combining his deep analysis with our result from Chapter 2, we describe a precise way to know whether or not a representation is primitive stable.

The other subspace we shall analyse is the *diagonal slice*. This slice corresponds to characters identified by triples of traces with the same complex value. If in the real representation case we were exploiting the easier nature of planar geom-

etry, here what helps us is symmetry: the peculiar configuration of the traces tells us that, in the Kleinian case at least, the quotient manifold is a genus 2 handlebody with a special rotational three-fold symmetry. Moreover, since any triple in the diagonal slice can be specified by giving just one complex parameter, it can also be plotted and visualised easily.

Firstly, in Chapter 5, after introducing the slice and explaining its symmetries, we apply our previous results about real representations to the real line and get a first overview.

In Chapter 6, finally, we deal with the rest of the slice. In [Min], the author shows that Schottky characters are primitive stable, and also that primitive stable representations form a strictly bigger set than Schottky characters. These last ones form an open set in the character variety, and are also the interior of the closed set of discrete and free representations. Our main result shows that there exists a rather large set of primitive stable representations which are not discrete nor free. The very special situation of the diagonal slice gives us a way of easily classifying the curves on the boundary of the handlebody which are invariant under the rotational symmetry. By using the machinery of pleating rays, we shall describe some of the points on the boundary of the discrete and free locus as endpoints of appropriate pleating rays, which correspond to bending these symmetric curves. Recent results obtained by Jeon and Kim about geometrically infinite representations in the boundary will finally complete the picture, and lead us to the main result.

## Chapter 2

# $\mathcal{BQ}$ and $\mathcal{PS}$

The main definitions and concepts are presented here. The reader who is well acquainted with the topic may want to skip this chapter. For completeness, we give a proof of the fact that the set  $\mathcal{PS}$  is contained in  $\mathcal{BQ}$ : it is a very straightforward conclusion, but apparently it nowhere appears in the literature. We also state two more sufficient conditions to conclude that a representation is primitive stable (Proposition 2.11 and Corollary 2.14), since we shall use them extensively later on.

### 2.1 Free groups and primitive words

A *free group of rank 2* can be defined by the following standard presentation:

$$F_2 = \langle a, b | \cdot \rangle.$$

Namely, it is the group generated by two elements, say  $a$  and  $b$ , with no relations. Probably the easiest way to think of it, as a set, is as the set of words in the alphabet  $a, b, A$  and  $B$ , where  $A$  and  $B$  denote, respectively, the inverses of  $a$  and  $b$ . For this reason, we will often use the term *word* to describe an element of the free group. The identity element is the empty word, made of no letters. The group operation is simply the juxtaposition of words. A word where no letter followed by its inverse appears, is called *reduced*. If the first letter of a reduced word differs from the inverse of the last letter, then we say that the word is *cyclically reduced*.

The *length* of a word, with respect to the generators  $a$  and  $b$ , is the number of letters that the word is formed by, after reduction. The length of the empty word is zero. A *generating pair*, or a *basis*, for  $F_2$  is a (unordered) pair of elements  $(g_1, g_2) \in F_2 \times F_2$  such that  $F_2 = \langle g_1, g_2 | \cdot \rangle$ . We say that a word is *primitive* if it belongs to some

generating pair. Primitive words will be crucial for us. In a free group of rank 2, primitive words can be written in a very special form, that will be very important later on. The typical pattern of a primitive word in  $F_2$  is described in the following theorem.

**Theorem 2.1** ([CMZ]). *Suppose that some conjugate of  $v = a^{n_1}b^{m_1} \dots a^{n_q}b^{m_q}$  and some conjugate of  $w = a^{\alpha_1}b^{\beta_1} \dots a^{\alpha_p}b^{\beta_p}$  form a basis of  $F_2$ , where  $p, q \geq 1$  and all of the indicated exponents are non-zero. Then, modulo trivial changes of notation (the possible replacement of  $a$  by  $a^{-1}$  or  $b$  by  $b^{-1}$ , throughout), there are integers  $n > 0$  and  $\varepsilon = \pm 1$  such that either:*

$$m_1 = \dots = m_q = \varepsilon\beta_1 = \dots = \varepsilon\beta_p = 1, \{n_1, \dots, n_q, \varepsilon\alpha_1, \dots, \varepsilon\alpha_p\} = \{n, n+1\}$$

or, symmetrically,

$$n_1 = \dots = n_q = \varepsilon\alpha_1 = \dots = \varepsilon\alpha_p = 1, \{m_1, \dots, m_q, \varepsilon\beta_1, \dots, \varepsilon\beta_p\} = \{n, n+1\}.$$

**Remark 2.2.** *It is worthwhile making a crucial observation here: by the theorem above, a cyclically reduced primitive word can only have a very special pattern:*

- *only two of the symbols from the alphabet  $\{a, b, A, B\}$  appear,*
- *one of the two always occurs as an isolated letter, i.e. with exponent either always equal to 1 or always equal to  $-1$ ,*
- *the powers of the other letter always have exponents belonging to  $\{K, K+1\}$ , where  $K$  can be any integer number.*

*One could actually go further and prove that the patterns we see in primitive words in  $F_2$  actually form what is called a balanced sequence, but this goes beyond the purposes of this thesis. The interested reader can find more details in [Ser].*

Since we will need a few more definitions about words in the free group later, it is worth giving them here: we say that a word is *primitive-blocking* if it never occurs as a subword of a cyclically reduced primitive word in  $F_2$ . We say that a word is *blocking* if one of its powers is primitive blocking.

**Example 2.3.** *The word  $w(a, b) = ab^2a$  is blocking, since its second power  $ab^2a^2b^2a$  cannot occur as a subword of a primitive word (given that both  $a$  and  $b$  are not always isolated). In other words,  $ab^2a^2b^2a$  is a primitive blocking word.*

Checking that an element in a free group is blocking is not always a simple task, and one efficient technique to determine this property is to look at the associated Whitehead graph. We shall explain later on what these graphs are and how they can help us.

## 2.2 Character variety

Since we shall study representations of the free group of rank 2 into the group of isometries of the hyperbolic space, it is convenient to consider the set of all these maps. As usual, we want to distinguish between two such maps only if they significantly differ, namely, if their images are not just conjugate. For this reason, the natural space to consider is the *character variety* defined as:

$$\chi = \text{Hom}(F_2, \mathbb{P}SL(2, \mathbb{C})) // SL(2, \mathbb{C}),$$

where the quotient has to be intended in the geometrical sense, that is classes in the quotient are closures of orbits under the action of  $SL(2, \mathbb{C})$  via conjugacy.

It is well known that there exists a bijection between points in  $\chi$  and triples in  $\mathbb{C}^3$ . This correspondence is given by assigning to a representation  $\rho \in \text{Hom}(F_2, \mathbb{P}SL(2, \mathbb{C}))$  the triple of traces  $(\text{tr}(\rho(a)), \text{tr}(\rho(b)), \text{tr}(\rho(ab))) \in \mathbb{C}^3$ . Since traces are invariant under conjugation, and representations with the same associated triple must be conjugate, this settles a bijection between points in  $\chi$  (i.e. classes of representations) and points in  $\mathbb{C}^3$ . In what follows, we will often deal with a fixed representation instead of its class in  $\chi$ ; the reader will immediately realise that all the definitions and concepts involved are invariant under conjugacy.

## 2.3 Bowditch conditions

Bowditch conditions (often shortened to BQ conditions) were introduced by Bowditch in [Bow], and analysed in deeper details in [TWZ]. The definition we will make use of is the following:

**Definition 2.4** (BQ conditions). *We say that  $\rho: F_2 \rightarrow \mathbb{P}SL(2, \mathbb{C})$  satisfies the BQ conditions if*

- *the image of every primitive element is a loxodromic isometry and*
- *the inequality*

$$|\text{tr}(\rho(w))| \leq 2$$

holds only for a finite number of (conjugacy classes of) elements of  $F_2$ .

**Remark 2.5.** *Observe that the two conditions above are invariant under conjugacy in  $\mathbb{PSL}(2, \mathbb{C})$ , hence it makes sense to speak about classes in the character variety satisfying these requirements. The second BQ condition, in particular, implies that the absolute values of traces of primitive elements do not accumulate anywhere in  $\mathbb{R}$ , that is we can replace the 2 in the inequality with any positive constant. This result has been proved by Bowditch and follows from Theorem 2 in [Bow]. This fact is highly non trivial, and the original argument makes use of the notion of Fibonacci growth for certain maps associated to Markoff triples. This machinery involves a number of topics and technicalities which go beyond the purpose of this paper. The interested reader can find more details in [Bow] and [TWZ], for instance.*

Denote by  $\mathcal{BQ}$  the subspace of the character variety where the BQ conditions hold.

## 2.4 Primitive stability

### 2.4.1 The map $\tau_{\rho, O}$

Let  $\rho$  be a homomorphism from  $F_2$  to  $\mathbb{PSL}(2, \mathbb{C})$ ,  $O$  a point in the hyperbolic space  $\mathbb{H}^3$  and  $C(F_2)$  the Cayley graph of  $F_2$  relative to a fixed pair of generators, equipped with the standard word metric (each edge has length one). We denote by  $\tau_{\rho, O}$  the unique  $\rho$ -equivariant map from  $C(F_2)$  to  $\mathbb{H}^3$ , sending the vertex  $w$  to  $\rho(w) \cdot O$  and the edge  $(w_1, w_2)$  to the geodesic segment between  $\rho(w_1) \cdot O$  and  $\rho(w_2) \cdot O$ . We will need often to label the edges of the Cayley graph, and similarly of its image under the map  $\tau_{\rho, O}$ , in order to keep track of paths in the graph. If  $(g, h)$  is an edge in  $C(F_2)$ , we assign to it the label  $g^{-1}h$ . We use this peculiar labelling convention to keep the labelling system “equivariant” under the action of the group. For, if  $g = e_1 \dots e_n$  and  $h = e_1 \dots e_n e_{n+1}$ , where  $e_i \in \{a, b, A, B\}$ , then the edge  $(g, h)$  is labelled by  $e_{n+1}$ . Consistently, the edge  $g^{-1} \cdot (g, h) = (\mathbb{1}, e_{n+1})$  is also labelled by  $e_{n+1}$ . Figure 2.1 shows the labelling convention for a portion of the Cayley graph.

### 2.4.2 Bi-infinite geodesics and uniformity

Consider a cyclically reduced primitive element  $w \in F_2$  and associate to its conjugacy class a bi-infinite word  $\bar{w} = \dots w w w \dots$ . If we think about  $F_2$  as the fundamental group of a wedge of two oriented circles with labels  $a$  and  $b$ , it is easy to see that  $\bar{w}$  determines a bi-infinite path in the bouquet of circles. Denote by  $\eta_w$  the lift of this



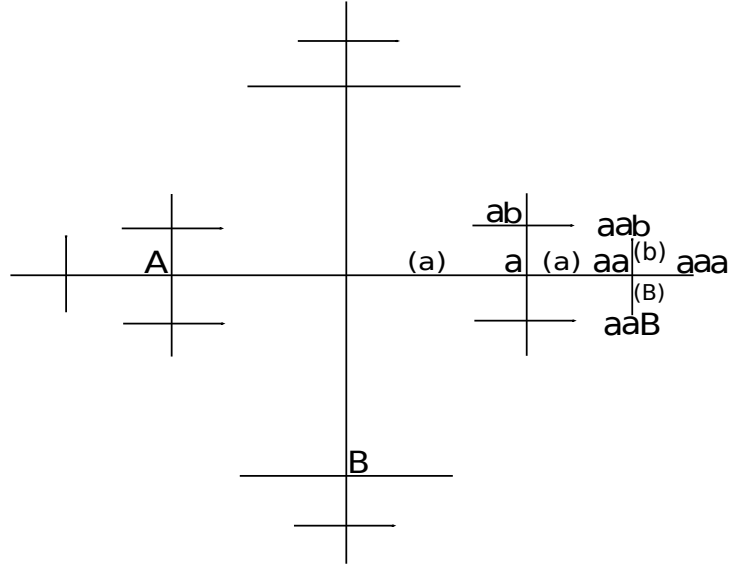


Figure 2.1: Edge labels are in brackets.

path to the universal cover of the wedge, namely the Cayley graph of  $F_2$ .  $\eta_w$  is a family of arcs in  $C(F_2)$ , and its image via  $\tau_{\rho,O}$  is a family of broken geodesics in the hyperbolic space, that we denote by  $G_w$ .

**Definition 2.6** (Quasi geodesics). *Let  $\gamma = \dots, v_{j-1}, v_j, v_{j+1}, \dots$  be a curve in the Cayley graph of  $F_2$ , that is an infinite sequence of adjacent vertices in the graph (note that, to avoid repetitions in the sequence, we do not consider a vertex to be adjacent to itself). Let  $\hat{\gamma} = \tau_{\rho,O}(\gamma)$ . We say that  $\hat{\gamma}$  is a  $(C, \varepsilon)$ -quasi-geodesic if for any two vertices of  $\gamma$ , say  $v_j, v_{j+h}$  we have:*

$$\frac{1}{C}h - \varepsilon \leq d_{\mathbb{H}}(\tau_{\rho,O}(v_j), \tau_{\rho,O}(v_{j+h})) \leq Ch + \varepsilon \quad (2.1)$$

*We say that  $\hat{\gamma}$  is a quasi geodesic if there exist parameters  $C > 1$  and  $\varepsilon \geq 0$  such that  $\hat{\gamma}$  is a  $(C, \varepsilon)$ -quasi-geodesic.*

*Note that  $h = d_C(v_j, v_{j+h})$ , is the length of the path from  $v_j$  to  $v_{j+h}$  according to the word metric on the Cayley graph.*

**Definition 2.7** (Primitive stability). *We say that  $\rho \in \text{Hom}(F_2, \mathbb{PSL}(2, \mathbb{C}))$  is a primitive-stable representation if there exist uniform constants  $C \geq 1, \varepsilon \geq 0$  such that each broken geodesic in  $G_w$  is a  $(C, \varepsilon)$ -quasi-geodesic, for any primitive  $w$  in  $F_2$ .*

We stress the fact that the constants  $C$  and  $\varepsilon$  do not depend on  $w$ , but they do depend on the choice of the basepoint  $O$  when defining the map  $\tau$ . Nevertheless, if such constants exist for a certain basepoint  $O$ , then they exist for any other basepoint, although they may be different. We denote by  $\mathcal{PS}$  the subset of the character variety where the property of primitive stability holds. The set  $\mathcal{PS}$  was introduced and analysed for the first time by Minsky in [Min]. In the paper, the author proves several results that we will extensively use. For the moment it is enough to know that  $\mathcal{PS}$  is an open set, and it (strictly) contains the set of Schottky (i.e. discrete, finitely generated, geometrically finite and purely loxodromic) representations ([Min], Lemma 3.2), which is in turn the interior of  $\mathcal{DF}$ , the (closed) set of discrete and faithful representations.

## 2.5 Sufficient conditions for primitive stability

### 2.5.1 $\mathcal{PS} \Rightarrow \mathcal{BQ}$

Here we prove that the set of primitive stable representations is contained in the set of representations satisfying the BQ conditions.

**Remark 2.8.** *Suppose we are given a representation  $\rho$  and we want to check whether it is primitive stable. Once a map  $\tau_{\rho,O}$  is fixed, we can always consider a special class of broken geodesics in  $\mathbb{H}^3$  which are part of the image of the Cayley graph under  $\tau_{\rho,O}$ . These are the broken geodesics passing through the basepoint  $O$ . More precisely, given a word  $w \in F_2$ , we take the (bi-infinite) path in the Cayley graph which, when looked at from the identity vertex, reads  $w$  in one direction and  $w^{-1}$  in the other. The image of this path is the broken geodesic that we want to consider. Note that, since conjugating a quasi geodesic by an isometry does not change its parameters, checking primitive stability on this set of lines is equivalent to check the whole set of lines in the image of the Cayley graph.*

**Proposition 2.9.**  $\mathcal{PS} \subseteq \mathcal{BQ}$ .

*Proof.* Let  $\rho \in \mathcal{PS}$ . The first BQ condition is automatically verified, as the orbit of, say, the origin under the action of an elliptic or a parabolic element clearly does not describe a quasi geodesic, therefore images of primitive elements must be loxodromic isometries. Now, fix a primitive element  $w$  and consider the situation shown in Figure 2.2 where the curve  $\gamma$  is the broken geodesic which is the image under the map  $\tau_{\rho,O}$  of the path in the Cayley graph corresponding to  $w$  and reading  $www \dots$  when starting from the identity vertex. The North-South dynamic generated by  $\rho(w)$

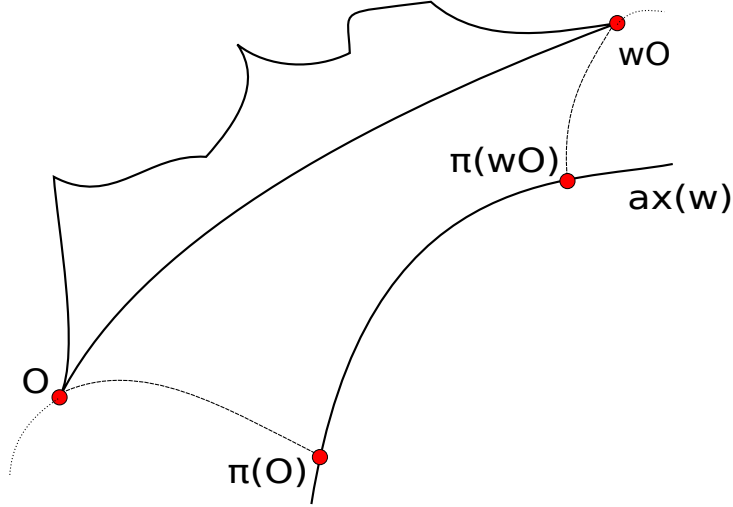


Figure 2.2: Projecting down the axes.

assures that the endpoints of  $\gamma$  and of the axes of  $\rho(w)$  coincide in the boundary at infinity. Moreover, by primitive-stability,  $\gamma$  is a  $(C, \varepsilon)$ -quasi-geodesic for some uniform parameters  $C$  and  $\varepsilon$  not depending on  $w$ . This implies the existence of a uniform constant  $K$  (depending only on  $C$  and  $\varepsilon$ ) such that  $\gamma$  and the axis of  $\rho(w)$  have Hausdorff distance less than  $K$ . Consequently, there exists a constant  $D > 0$  such that if  $p$  and  $q$  are points on  $\gamma$ , then their hyperbolic distance is less than or equal to the distance between their nearest point projections (onto the axes) times  $D$ , up to an additive constant. By definition of  $(C, \varepsilon)$ -quasi-geodesic, if  $\|w\|$  denotes the word length of  $w$ , then  $\|w\| \leq Cd_{\mathbb{H}}(O, w \cdot O) + \varepsilon$  for any primitive word  $w$ . We denote the *translation length* of an isometry  $S \in \mathbb{PSL}(2, \mathbb{C})$  by  $l(S)$ . Note that the translation length and the trace of the matrix of  $S$  are related by an explicit formula (see Section 5.1 for the exact expression). Putting everything together we get:

$$\|w\| \leq Cd_{\mathbb{H}}(O, w \cdot O) + \varepsilon \leq CDl(\rho(w)) + \varepsilon.$$

Since there are only finitely many primitive elements with word length bounded above by a constant, this proves that the traces of their images cannot accumulate anywhere on the positive real semi-line.  $\square$

**Remark 2.10.** *We want to stress the fact that, for our purposes,  $\mathcal{BQ}$  conditions will always be the ones described in Definition 2.4. A more relaxed definition, generally named as extended Bowditch conditions, is used in the literature, especially when dealing with identities à la McShane, involving converging sums on simple closed*

curves on the torus, and generalisations. Extended Bowditch conditions allow primitive elements in  $F_2$  to have images which are parabolics in  $\mathbb{PSL}(2, \mathbb{C})$ . Naturally, these representations cannot belong to  $\mathcal{PS}$ , due to the dynamic nature of parabolic isometries. Hence we would have representations which are in the extended  $\mathcal{BQ}$  set but are not primitive stable. This phenomenon justifies our choice to deal only with the more classic Bowditch conditions, that give us at least one inclusion ( $\mathcal{PS} \subseteq \mathcal{BQ}$ ) and makes the whole picture neater.

### 2.5.2 Two more conditions

We discuss here two sufficient conditions for primitive stability, as we shall make use of them further on. The first result has been proved by Minsky. We just copy this specific statement here:

**Proposition 2.11** ([Min], Lemma 3.2, statement (1)). *Schottky characters are primitive stable.*

The assumption of the proposition above can be relaxed to obtain a stronger statement, expressed in Corollary 2.14, where we allow the character to be not purely loxodromic. The main result we need in order to prove this, is the following proposition due to Floyd ([Flo], p.216)

**Proposition 2.12** ([Flo]). *Let  $G$  be a discrete, finitely generated, geometrically finite subgroup of  $\mathbb{PSL}(2, \mathbb{C})$  with no parabolics, and let  $O \in \mathbb{H}^3$  be a fixed basepoint. Then there exists  $C > 0$  such that:*

$$\frac{\|g\|}{C} \leq d_{\mathbb{H}}(O, g \cdot O) \leq C\|g\|, \quad \forall g \in G \quad (2.2)$$

*If the image contains parabolic elements instead, then there exist  $C, k > 0$  such that:*

$$\frac{2\log(\|g\|)}{C} - k \leq d_{\mathbb{H}}(O, g \cdot O) \leq C\|g\|, \quad \forall g \in G \quad (2.3)$$

In our case we have to think of  $G$  as the image of a representation of  $F_2$ . We will also need an important statement whose proof is contained in the argument used by Minsky in [Min], Theorem 4.1:

**Proposition 2.13.** *Let  $\rho$  be discrete and geometrically finite, and let  $N = \mathbb{H}^3/\rho(F_2)$  be the quotient hyperbolic manifold. Assume that all cusps in the quotient manifold correspond to blocking elements. Then there exists a compact subset  $K \subseteq N$ , containing the convex core, such that every geodesic corresponding to a primitive element of  $F_2$  is contained in  $K$ .*

*Proof.* See proof of Theorem 4.1 in [Min] for a proof of the statement.  $\square$

Theorem 4.1 in [Min] assumes that the quotient manifold has a unique blocking cusp, but in reality the same argument is valid even when the cusps are more than one. This is further discussed in Remark 6.9 later on in the thesis.

The proof of the next corollary follows the line of Floyd's proof of Proposition 2.12, and makes use of Proposition 2.13 above, extending the result stated in Proposition 2.11.

**Corollary 2.14.** *Let  $\rho$  be discrete, geometrically finite and such that the only parabolic elements, if any, are blocking. Then  $\rho$  is primitive stable.*

*Proof.* First of all, note that, for any pair of elements in the free group  $w_1, w_2$ , we have:

$$d_{\mathbb{H}}(\rho(w_1) \cdot O, \rho(w_2) \cdot O) = d_{\mathbb{H}}(O, \rho(w_1^{-1}w_2) \cdot O).$$

If there were no parabolic elements in the group, then the inequalities (2.2) would give us the desired result with  $g = \rho(w_1^{-1}w_2)$ , but a priori only (2.3) holds. We claim that the left hand side inequality in (2.3) can be replaced with the stronger left hand side inequality in (2.2) if we only consider words which are primitive.

Let  $D$  be a finite sided fundamental polyhedron for the group  $G = \rho(F_2)$ , and denote by  $\mathcal{N}$  the hyperbolic convex hull of the limit set. Let  $K$  be the compact set containing all geodesics corresponding to primitive elements in the quotient manifold, as in Proposition 2.13, and let  $\hat{K}$  be its lift in  $\mathbb{H}^3$ . For a positive value  $\varepsilon$ , call  $\mathcal{M}_{thin}(\varepsilon)$  the  $\varepsilon$ -thin part of the quotient, that is the set of points where the injectivity radius is smaller than  $\varepsilon$ . Call  $\mathcal{H} = \mathcal{H}_{\varepsilon}$  its lift to the universal cover. We fix a value of  $\varepsilon$  small enough to guarantee that:

- $\mathcal{H}$  is a  $G$ -invariant set of disjoint horoballs, and
- $\mathcal{H} \cap \hat{K} = \emptyset$ .

We shall only consider the complement  $\mathcal{V} = \mathbb{H}^3 \setminus \mathcal{H}$ . Assume, without loss of generality, that the origin  $O$  belongs to the intersection between  $D$  and  $\hat{K}$ . Set  $D' = D \cap \hat{K}$ . Since  $D'$  is compact, its diameter  $d$  is finite. Set

$$L = \max_{w \in F_2} \{ \|w\| \mid d_{\mathbb{H}}(O, \rho(w) \cdot O) < 1 + 2d \}.$$

Now, take a  $g \in G$  which is not blocking, and subdivide the segment between  $O$  and  $g \cdot O$  into intervals of length 1, allowing the last interval to be shorter. Join each division point to the closest point in the  $G$ -orbit of  $O$ . We claim that the

line between  $O$  and  $g \cdot O$  is all contained in  $D'$ . For,  $g$  being non blocking, we can extend it on the right to a minimal primitive word, say  $g_p$ . The axis of  $g_p$  lies in  $\hat{K}$ , since geodesics corresponding to primitive words cannot exit the compact set  $K$ . By Proposition 2.13, the broken line joining the closest orbit points lies uniformly close to the axis of  $g_p$ , and so does the segment  $[O, g \cdot O]$ . It follows that all the segment we are considering lies in  $\hat{K}$ , and never intersects the set  $\mathcal{H}$ . The joining lines from the division points to the origin orbit travel from one copy of  $D'$  to the adjacent one: thus they must all have length less than or equal to  $2d$ . Consequently, when moving from a division point to the next one along the geodesic segment between  $O$  and  $g \cdot O$ , then along the joining line to the closest point in the orbit of the origin, and finally back to the geodesic line along the same joining segment, we never travel for a distance greater than  $1 + 2d$ . It follows that  $\|g\| \leq d_{\mathbb{H}(O, g \cdot O)} \cdot L + 1$ , giving the required inequality for  $g$ . Suppose that  $w_1$  and  $w_2$  are primitive words in  $F_2$ . Since the Cayley graph of a free group is a tree, any two of its vertices can be joined by a unique path. Suppose, without loss of generality, that  $w_1 = e_1 \dots e_n$  and  $w_2 = e_1 \dots e_n e_{n+1} \dots e_{n+h}$ , where  $e_i \in \{a, b, A, B\}$ . Then  $w_1^{-1}w_2 = e_{n+1} \dots e_{n+h}$ , and in particular is a subword of  $w_2$ , hence not blocking, hence not parabolic. Setting  $g = \rho(w_1^{-1}w_2)$  we get our result.  $\square$

Many conjectures have been made regarding possible relations between primitive stability and Bowditch conditions.

Bowditch conjectured that in the slice of type preserving representations (i.e. characters such that  $\text{tr}(\rho(abAB)) = -2$ ), BQ conditions hold if and only if the representation is quasi Fuchsian.

As we have shown, it is easily seen that  $\mathcal{PS} \subseteq \mathcal{BQ}$ . Whether the two sets actually coincide is another open conjecture. In what follows, we obtain partial results in this direction, that seem to confirm the validity of the supposition.

## Chapter 3

# Primitive stability and tessellations of the hyperbolic plane

It is well known that the subgroup  $\mathbb{P}SL(2, \mathbb{R}) \subseteq \mathbb{P}SL(2, \mathbb{C})$  can be identified with the (orientation preserving) isometries of the hyperbolic plane. When the image of a representation  $\rho$  is contained in  $\mathbb{P}SL(2, \mathbb{R})$ , primitive stability can be proven by some more concrete machinery, since the geometry of the hyperbolic disc allows more “bare hands” constructions than the geometry of the three dimensional hyperbolic space.

Suppose for a moment that we are given a representation  $\rho: F_2 \rightarrow \mathbb{P}SL(2, \mathbb{R})$  which is discrete (in this case we say that  $\rho(F_2)$  is a *Fuchsian group*), so that, in particular, the quotient of the hyperbolic plane under the action of  $\rho(F_2)$  is a hyperbolic orbifold. By classic results, one can always find a finite-sided fundamental domain for the action of  $G = \rho(F_2)$ , that is a finite-sided polygon that contains exactly one representative for each orbit. The copies of this region under the action of  $G$  *tessellate* the entire plane, leaving no gaps and never overlapping. Of course, the situation changes as soon as the representation ceases to be discrete, and primitive stability does not always hold. In what follows, we will show that, in order to prove primitive stability for a given representation, we can still proceed by similar arguments. The main idea is that some property of the commutator matrix will allow us to find some “special” polygon which will play the same role as the fundamental domain; as we act by primitive words on this region, we lay down copies of it in a “non-chaotic” manner, without overlapping or zig-zagging too much. The precise statements below will clarify the argument.

**Remark 3.1.** *The techniques explained below are developed “ad-hoc” for subgroups of  $\mathbb{PSL}(2, \mathbb{R})$ , since the geometry of the hyperbolic plane is easier to understand than that of the hyperbolic space. Nevertheless, the results presented in this chapter extend obviously to any group in  $\mathbb{PSL}(2, \mathbb{C})$  which can be conjugated into  $\mathbb{PSL}(2, \mathbb{R})$ . We shall see that in most of the situations that we will deal with, this will indeed be the case.*

We say that  $\rho: F_2 \rightarrow \mathbb{PSL}(2, \mathbb{R})$  is a representation with intersecting axes if the images of a pair of generators (hence any) are hyperbolic isometries whose axes are distinct and cross. If  $S, T \in \mathbb{PSL}(2, \mathbb{R})$ ,  $[S, T] = STS^{-1}T^{-1}$  encodes information about the geometry of the elements  $S$  and  $T$ . For a fixed representation  $\rho$ , we denote by  $\tau = \tau(\rho)$  the trace of the matrix corresponding to the commutator of a generating pair. The following lemma is stated and proved in [Mat] (Lemma 3.2.2).

**Lemma 3.2.** *Suppose that  $S$  and  $T$  are hyperbolic isometries of the hyperbolic plane, and denote by  $ax(S)$ ,  $ax(T)$  their axes. Suppose also that their axes are distinct. Then  $ax(S) \cap ax(T) \neq \emptyset$  if and only if  $\tau = \text{tr}([S, T]) < 2$ .*

**Corollary 3.3.** *If a pair of generators has crossing axes, then so does any other pair*

In other words,  $\rho$  is a representation with intersecting axes if and only if  $\tau(\rho) < 2$ . The restriction  $\tau \in (-\infty, 2)$  still allows the commutator to be hyperbolic ( $\tau \in (-\infty, -2)$ ), parabolic ( $\tau = -2$ ) and elliptic ( $\tau \in (-2, 2)$ ). In the first two cases, the representation is discrete ([Gil]), and therefore we are in the case described in Corollary 2.14, and  $\rho$  is primitive stable. If the commutator is elliptic, the representation might or might not be discrete. In [Gil], the author describes a complicated algorithm and gives a precise criterion to understand how the geometry of  $\rho(a)$  and  $\rho(b)$  determines the discreteness of the representation.

Regardless of discreteness, in this chapter we prove the following proposition:

**Proposition 3.4.** *Let  $\rho: F_2 \rightarrow \mathbb{PSL}(2, \mathbb{R})$  be a representation with intersecting axes. Then  $\rho$  is primitive stable.*

In what follows, for simplicity, we will write  $A = \rho(a)$  and  $B = \rho(b)$ , using capital letters for matrices, or isometries, in  $\mathbb{PSL}(2, \mathbb{R})$ , when no confusion can arise. Below we give a proof of our main statement. The idea is roughly the following: a representation with intersecting axes is always a holonomy of some geometrical structure on a torus with one boundary component. This allows us to find some



embedded polygon with side-pairings which bounds a disc in the hyperbolic plane. When the commutator is elliptic, the polygon is a quadrilateral where opposite sides are paired by (images of) suitable generators of  $F_2$ . At this point, we use a simple condition on the angles of this quadrilateral to carry on a geometrical argument that concludes the proof. More precisely, we shall use primitive words to lay down adjacent copies of the quadrilateral, as one would normally do when tiling a space with copies of a fundamental region. The simple shape of our “fundamental region” and some observations about primitive words, will allow us to produce chains of hyperplanes (geodesics) for each broken geodesic  $\gamma_w$  arising from a primitive word  $w = w(a, b)$ . The hyperplanes will be arranged in such a way that will show that  $\gamma_w$  is a quasi geodesic with parameters that do not depend on  $w$ , proving primitive-stability for  $\rho$ . Let us see carefully how to do that. We have already observed that  $\tau$  has to belong to the semi-line  $(-\infty, 2)$  and that  $\tau \in (-\infty, 2]$  implies discreteness, hence primitive stability by Corollary 2.14. We can thus assume that  $\tau \in (-2, 2)$ , that is the commutator is elliptic.

Recall that  $F_2$  can be thought of as the fundamental group of a topological torus with one open disc removed, that we denote by  $S_1^1$ . In [Mat] (Theorem A), the author proves that a representation  $\rho: \pi_1(S_1^1) \rightarrow SL(2, \mathbb{R})$  is the holonomy of (at least) one cone-manifold structure on  $S_1^1$  if and only if  $\rho$  is not virtually abelian. Luckily for us, virtually abelian representations cannot have  $\tau < 2$ , so that representations with intersecting axes are never virtually abelian. Therefore, our  $\rho$  induces some cone-manifold structure on  $S_1^1$ . It is easily seen that if the commutator is elliptic, then  $S_1^1$  with the induced structure is a torus with a unique cone-point with angle less than  $2\pi$ . Following [Mat] (Lemma 5.1.5), whenever we are dealing with a holonomy of a cone-manifold structure, we can find an immersed pentagon in  $\mathbb{H}^2$ , which is described by the cycle:

$$p \rightarrow B'p \rightarrow A'B'p \rightarrow B'^{-1}A'B'p \rightarrow [A'^{-1}, B'^{-1}]p \rightarrow p,$$

where  $p$  is a generic point in  $\mathbb{H}^2$  and if necessary we change the generating pair  $(a, b)$  to another pair  $(a', b')$ . Note that changing generating pairs does not affect the geometry of a representation: the trace of the commutator is preserved under changes of basis, so that

$$tr([A, B]) = tr([A', B']) = \tau \in (-2, 2).$$

In particular, the axes of the new generators still intersect in one point, the new commutator is still elliptic and it rotates about its fixed point by the same angle.

For this reason, with a little abuse of notation, we shall denote the new pair again with the symbols  $a$  and  $b$ . Let  $p$  be the fixed point of the commutator. The relation  $[A, B] \cdot p = p$  tells us that the pentagon we had before now degenerates to a quadrilateral with opposite sides paired by  $A$  and  $B$  and such that the sum of the four internal angles is equal to the angle by which the commutator rotates about  $p$ . Also, since opposite sides are paired by isometries, and hence are of equal lengths, opposite angles have the same amplitude, too. By [Gol2], Lemma 3.4.2, the polygon described above is embedded, that is the four geodesic segments obtained by moving the fixed point by hyperbolic isometries are pairwise disjoint and bound a quadrilateral. Figure 3.1 depicts the relative position of the axes, the fixed point and the quadrilateral. Here  $a_A, r_A, a_B, r_B$  indicate the attracting and repelling fixed points of  $A$  and  $B$ . We can summarise the progress that has been made so far

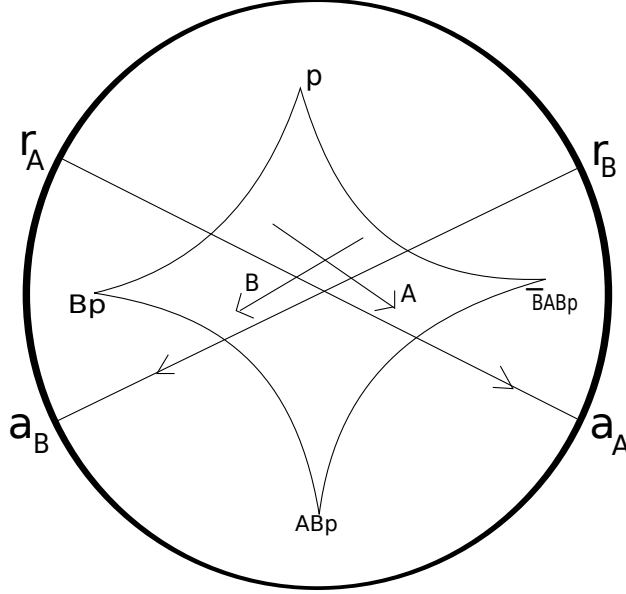


Figure 3.1: The intersecting axes and the quadrilateral.

in the following way:

given a representation with intersecting axes and elliptic commutator, we can always find an embedded quadrilateral bounding a disc in the hyperbolic plane, whose opposite sides are paired by the isometries corresponding to a pair of generators for  $F_2$ , and whose internal angles add up to the rotation angle of the commutator. Now it is time for a little lemma:

**Lemma 3.5.** *Suppose  $Q$  is a quadrilateral with side-pairing as the one depicted in Figure 3.1. Then the strip  $A^k \cdot Q$ ,  $k \in \mathbb{Z}$ , is entirely contained between the*

geodesic extensions of the sides paired by  $B$ . (Clearly an equivalent statement holds by swapping the roles of  $A$  and  $B$ .)

*Proof.* We shall use the notation shown in Figure 3.2.  $I$ ,  $L$ ,  $M$ , and  $N$  are the labels

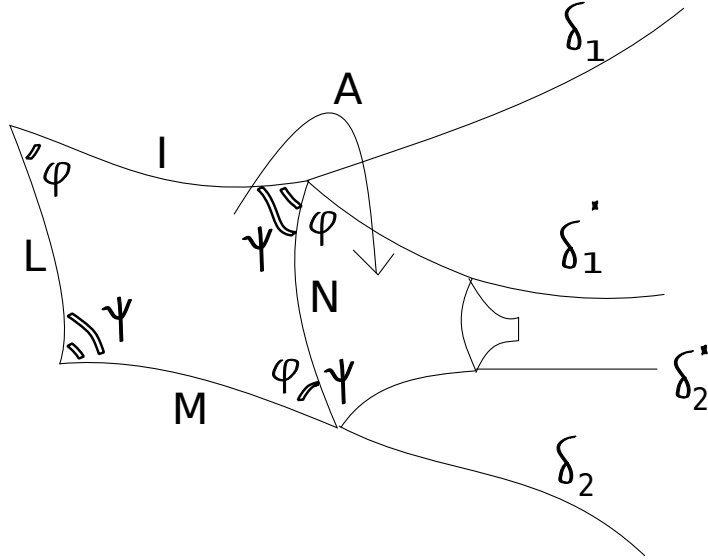


Figure 3.2: Part of the right branch of the strip of copies of  $Q$ .

of the sides of the quadrilateral.  $\delta_1$  and  $\delta_2$  are the geodesics containing  $I$  and  $M$ ,  $\varphi$  and  $\psi$  the internal angles at the vertices of the polygon. Consider the first adjacent copy  $A \cdot Q$ . The segments  $A \cdot I$  and  $A \cdot M$  have to lie between  $\delta_1$  and  $\delta_2$ . For, in hyperbolic geometry the sum of the internal angles of a quadrilateral is less than  $2\pi$ . Hence we immediately get

$$2(\varphi + \psi) < 2\pi \Rightarrow \varphi + \psi < \pi.$$

Moreover,  $\delta_1 \cap \delta_2 = \emptyset$ , so  $A \cdot I \cap A \cdot M = \emptyset$  as well. Thus the copy  $A \cdot Q$  is entirely contained between  $\delta_1$  and  $\delta_2$ .

If we repeat the argument we see that  $A^2 \cdot Q = A \cdot (A \cdot Q)$  is contained between the lines  $\delta'_i = A \cdot \delta_i$  ( $i = 1, 2$ ). On the other hand, two geodesics can intersect only once and  $\delta_i \cap \delta'_i$  is a vertex of  $Q$ . Therefore the whole semi-line contained in  $\delta'_i$  and starting from  $Q$  lies between  $\delta_1$  and  $\delta_2$ . This tells us that  $A^2 \cdot Q$  is not only contained between  $\delta'_1$  and  $\delta'_2$ , but also between  $\delta_1$  and  $\delta_2$ . By repeating the argument infinitely many times and applying it to the left side too (using now  $A^{-1}$  instead of  $A$ ), we conclude the proof.  $\square$

Now we continue with the proof of Proposition 3.4. Recall that a primitive word in  $F_2 = \langle a, b | \cdot \rangle$  is an element of the group which can be extended to a minimal free generating set. As an element of  $F_2$ , it can be thought of as a (cyclically reduced) word in the alphabet  $a, b, a^{-1}, b^{-1}$ . We shall exploit the specific patterns of words in  $F_2$  explained in Remark 2.2, to subdivide the word in pieces which can show a very limited number of sequences. From now on, for simplicity and without loss of generality, we suppose that we are given a word in the letters  $a$  and  $b$ ,  $w = w(a, b) = ab^{n_1}ab^{n_2} \dots ab^{n_p} = e_1 \dots e_n$ , where  $e_i \in \{a, b\}$ ,  $n = (\sum_i n_i) + p$ . The broken geodesic  $\gamma_w$  that we want to control is given by the sequence of vertices

$$\dots, e_n^{-1}p, p, e_1p, e_1e_2p, \dots$$

We can label each geodesic segment in  $\gamma_w$  with a letter, so that going along  $\gamma_w$  reads off  $\dots www \dots$ . As we start from  $p$  and we move towards the “right” branch of the line, we also lay down copies of the “fundamental polygon”  $Q$ , obtaining a strip. Going from one copy of  $Q$  to the next one, clearly reads off the same letter that we’d read by going along  $\gamma_w$ .

More precisely, the geodesic segment defined by the endpoints  $g \cdot p$  and  $ge_i \cdot p$  is labeled by  $e_i$ . Likewise, suppose we start with a copy of the original quadrilateral, say  $g \cdot Q$ , and the next letter we read in  $w$  is  $e_i$ . First, we bring the quadrilateral back to the original one by applying  $g^{-1}$ , then we move it by  $e_i$  and finally we move it back to be adjacent to  $g \cdot Q$ . Therefore, the next copy of  $Q$  that we lay down is  $(ge_i g^{-1})(g) \cdot Q$ , that is  $ge_i \cdot Q$ .

Figure 3.3 should help the reader to understand the situation. Consider the following sequence of hyperplanes. Whenever we read a  $b$ , we draw the hyperplane given by the geodesic extending the edge between the two adjacent regions. If we find an  $a$ , we skip that and we proceed drawing hyperplanes from the next  $b$ . Starting from  $p$  and moving forward, we also enumerate the hyperplanes as  $H_0, H_1, H_2, \dots$ . We claim that  $H_i \cap H_{i+1} = \emptyset$  for each  $i$ .

**Lemma 3.6.** *With the notation as above,  $H_i \cap H_{i+1} = \emptyset$  for each  $i$ .*

*Proof.* Firstly, we observe that between two consecutive hyperplanes  $H_i, H_{i+1}$ , we can only see two patterns:  $bb$  or  $bab$ . In the former case, the corresponding vertices of  $\gamma_w$  will be of the form  $gb \cdot p, gbb \cdot p$ , where  $g$  is some subword of a power of  $w$ . By applying  $g^{-1}$ , we move back to our standard configuration (shown in the left-hand side of Figure 3.4), where  $Q$  is the original quadrilateral we started from. It should be now clear that the two edge extensions  $H_i$  and  $H_{i+1}$  are just the extensions of two opposite sides of  $Q$ , hence they do not intersect.

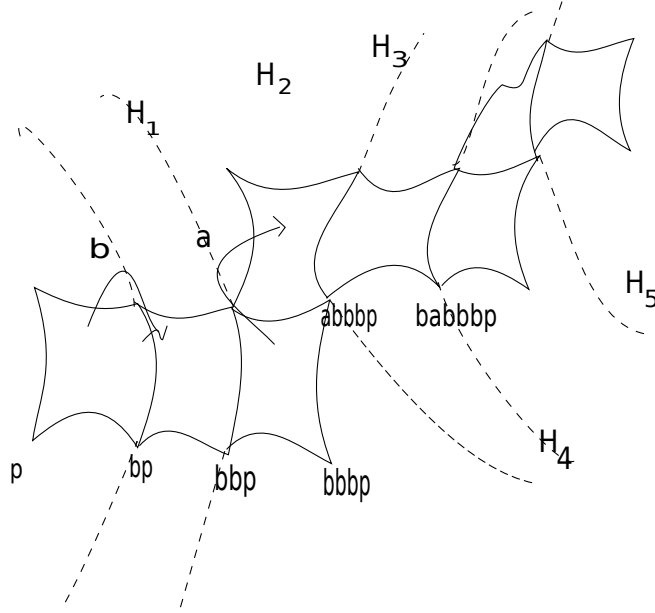


Figure 3.3: Laying down polygons as we move along  $\gamma_w$ .

If the pattern is  $bab$  instead, then we move everything back to  $p$  as above and we are in the situation depicted in the right-hand side of Figure 3.4. Here  $\eta_1$  and  $\eta_2$  are just the images of  $H_i$  and  $H_{i+1}$ , respectively, after translating back to the origin. We claim that  $\eta_1, \eta_2$  do not intersect in the hyperbolic plane, nor in its boundary. By Lemma 3.5, they should contain an infinite strip made of isometric copies of the fundamental region, hence they cannot cross at any point in  $\mathbb{H}^2$ . If we assume they meet at infinity, say at a point  $q \in \partial\mathbb{H}^2$ , the situation is slightly more delicate. Observe that  $\eta_2 = ab \cdot \eta_1$ . Consider the family  $\{l_k = ab^k \eta_1 | k \in \mathbb{Z}\}$ . They are all geodesic lines with one endpoint equal to  $q$ ; call the other endpoint  $q_k$ . If

$$\lim_{k \rightarrow \infty} q_k = \lim_{k \rightarrow -\infty} q_k = q,$$

then  $ab$  is a parabolic element, and this gives a contradiction, since the product of hyperbolic isometries with intersecting axes is always hyperbolic. Then, at least on one side,  $l_k$  accumulates to some limit line  $l$ . In particular, the Hausdorff distance between  $l_k$  and  $l$  goes to zero. Again by Lemma 3.5, this leads to a contradiction since if the lines were converging, the Hausdorff distance between two consecutive ones would become eventually less than the diameter of the quadrilateral. Thus  $H_i \cap H_{i+1} = \emptyset$  for each  $i$  and we are through.  $\square$

Note that, because of the way we draw the hyperplanes, each  $H_i$  separates

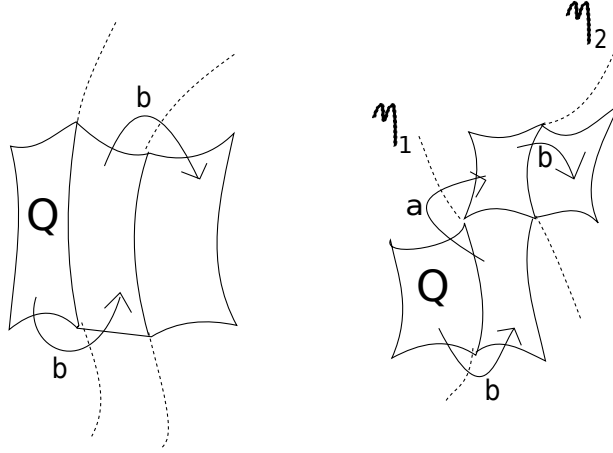


Figure 3.4: The only two possible patterns, centred at the origin.

the hyperbolic plane into two regions, one containing all the  $H_j$  with  $j < i$  and one containing all the  $H_j$  with  $j > i$ . Moreover, from Lemma 3.6, the distance between  $H_i$  and  $H_{i+1}$  has to be a positive number, and this number only depends on the pattern between the hyperplanes. Since these patterns are only two, namely  $bb$  and  $bab$ , we get only two possible distances between consecutive hyperplanes. Denote by  $\mu \in \mathbb{R}^+$  the smallest among the two.

Of course we could repeat the whole story for primitive words in  $a^{-1}$  and  $b$ , obtaining a new value  $\mu'$  and then choosing only the minimum between  $\mu$  and  $\mu'$ . In this case we assume that  $\mu$  is the minimum. Now, take any two vertices  $q_1$  and  $q_2$  along the line  $\gamma_w$ . Without loss of generality, we assume that they both belong to the positive branch. Thus they define a subword  $w'$  of some power of  $w$ . Set  $\|w'\| = n$ , where  $\|w'\|$  denotes the usual word length in the generators  $a$  and  $b$ . The geodesic segment  $[q_1, q_2]$  has to pass at least through  $n/2$  hyperplanes. Split  $[q_1, q_2]$  in subsegments according to the hyperplane intersection points. We see that

$$l[q_1, q_2] = d_{\mathbb{H}}(q_1, q_2) \geq \left(\frac{n}{2} - 1\right)\mu = \left(\frac{\mu}{2}\right)n - \mu.$$

Since  $\mu$  does not depend on the word  $w$ , we get uniform parameters for the lower inequality for quasi-geodesics. This concludes the proof of Proposition 3.4.

**Remark 3.7.** *Note that the two Lemmas above only make use of the fact that the considered words, once written in their cyclically reduced form, are made of two letters in the alphabet  $\{a, b, A, B\}$ , and that we never really make use of the fact that*

one of the two is isolated. Although we do not need it here, both lemmas could be naturally extended to the more general case, dropping the isolated letter assumption.

We finish the chapter by summarising what we have proved in the following theorem, which follows by combining Proposition 3.4 with Remark 3.1.

**Theorem 3.8.** *Let*

$$\rho: F_2 = \langle a, b | \cdot \rangle \rightarrow \mathbb{P}SL(2, \mathbb{C})$$

*be a representation, and assume that there exists  $g \in \mathbb{P}SL(2, \mathbb{C})$  such that*

$$g^{-1}\rho(F_2)g \subseteq \mathbb{P}SL(2, \mathbb{R}).$$

*Set  $\tau = \text{tr}([\rho(a), \rho(b)])$ . If  $\tau < 2$ , then  $\rho \in \mathcal{PS}$ .*

Observe that a group satisfying the hypotheses of the theorem above must be made only of matrices whose trace is real, but not all groups with only real traces arise by conjugating subgroups of  $\mathbb{P}SL(2, \mathbb{R})$ . In the next chapter, we shall extend our results to the bigger class of *real representations*.

## Chapter 4

# Real representations, $\mathcal{BQ}$ conditions and primitive stability

We now look at the subset of the character variety which is parametrised by triples of real numbers. A character corresponding to a triple of real numbers is called a *real representation*. Throughout the chapter, therefore, each representation will be assumed to be a homomorphism  $\rho: F_2 \rightarrow \mathbb{P}SL(2, \mathbb{C})$  such that the traces of the image matrices of a pair of generators and their product are all real. The trace of any other matrix in the image can be expressed as a polynomial with integer coefficients in these three traces, hence every matrix in the image must have real trace. We try to establish a criterion under which one can decide whether a given representation is primitive stable or not, and whether it belongs to the  $\mathcal{BQ}$  set.

### 4.1 Three real traces imply Fuchsian

In this section, we show that a real representation that sends generators to loxodromic elements is always conjugate to a representation into  $\mathbb{P}SL(2, \mathbb{R})$ . A sketch of the proof can be found in [MSW] 6.6; we present here a slightly more detailed argument:

**Proposition 4.1.** *Let*

$$\rho: F_2 = \langle a, b | \cdot \rangle \rightarrow \mathbb{P}SL(2, \mathbb{C}), \quad a \mapsto S, b \mapsto T.$$

*If  $\text{tr}(S), \text{tr}(T), \text{tr}(ST) \in \mathbb{R} \setminus [-2, 2]$ , then  $\rho$  can be conjugated into  $\mathbb{P}SL(2, \mathbb{R})$ .*



*Proof.* Let us start by conjugating by an element  $g \in \mathbb{P}SL(2, \mathbb{C})$  to put  $S$  in its standard form:

$$g^{-1}Sg = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

We now conjugate by another hyperbolic isometry  $h$  that rotates the hyperbolic space around the axis of  $g^{-1}Sg$  and moves one of the endpoints of  $g^{-1}Tg$  onto a real number. Write

$$h^{-1}g^{-1}Tgh = \begin{pmatrix} u & v \\ w & z \end{pmatrix}.$$

We know that  $\text{tr}(T)$  and  $\text{tr}(ST)$  are real, so that

$$u + z \in \mathbb{R} \text{ and } \lambda u + \lambda^{-1}z \in \mathbb{R}.$$

From the first condition we get  $u = x + iy$  and  $v = x' - iy$  for some  $x, y, x' \in \mathbb{R}$ . Using the second condition, we get  $iy(\lambda - \lambda^{-1}) \in \mathbb{R}$ . Since  $\lambda \neq \pm 1$ , it follows  $y = 0$ , hence  $u, z \in \mathbb{R}$ . The fixed points of  $h^{-1}g^{-1}Tgh$  are given by the formula

$$\frac{u - z \pm \sqrt{(\text{tr}T)^2 - 4}}{2w},$$

and we know that at least one of the two must be real. It follows that  $w$ , and hence  $v$ , must be real too.  $\square$

**Remark 4.2.** *The limit set of a Fuchsian group must be contained in the equatorial circle on the Riemann sphere. If a group is conjugate to a Fuchsian group, then its limit set is again contained in some geometrical circle on the Riemann sphere. In particular, its regular set is never empty. Under this assumption, checking that a group is Schottky is equivalent to check that it is free and purely loxodromic (see Theorem 4.23 in [MT]).*

## 4.2 The action of $\text{Out}(F_2)$

For a given representation  $\rho: F_2 = \langle a, b | \cdot \rangle \rightarrow \mathbb{P}SL(2, \mathbb{C})$ , set

$$x = \text{tr}(\rho(a)), \quad y = \text{tr}(\rho(b)), \quad z = \text{tr}(\rho(ab)).$$

The value of the trace of the commutator matrix depends on the three real numbers  $x$ ,  $y$  and  $z$  and can be easily computed by the polynomial identity:

$$\tau(x, y, z) = x^2 + y^2 + z^2 - xyz - 2. \quad (4.1)$$

We look at  $\tau$  as a real function defined on  $\mathbb{R}^3$ . Clearly, a point in  $\mathbb{R}^3$  also defines a conjugacy class of a representation of the free group of rank 2, and identify  $\mathbb{R}^3$  with the slice of the character variety corresponding to real representations.

There is a natural action of the group  $Out(F_2)$  on the character variety, defined by:

$$(g \cdot \rho)(w) = \rho(g(w)).$$

By the observation above, the action can be thought of as an action on  $\mathbb{R}^3$ , and it is commensurable with the action of the group of automorphisms of the polynomial 4.1. This group, that we denote by  $\Gamma$ , is isomorphic to  $PGL(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ . In [Gol2], the author computes the generators of this group, studies its action on the character variety and obtains some results that we shall mention below.

We let a real parameter  $t$  free to vary across the real line, and distinguish between the following cases:

1.  $t \in (-\infty, -2)$ ,
2.  $t = -2$ ,
3.  $t \in (-2, 2)$ ,
4.  $t = 2$ ,
5.  $t \in (2, 18]$ , and
6.  $t \in (18, \infty)$ .

In what follows, we voluntarily ignore the difference between elements of the character variety and triples of real numbers. Moreover, we shall often say just “representation” instead of “class of representations”, since all the features we are interested in are invariant under conjugation.

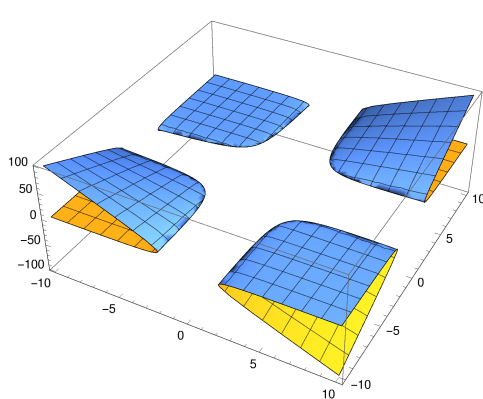
For each of the cases, we pick an arbitrary value of  $t$  in the interval of interest and plot the preimage  $\tau^{-1}(t)$  or, equivalently, the zeroes of the polynomial

$$p_t(x, y, z) = x^2 + y^2 + z^2 - xyz - (t + 2).$$

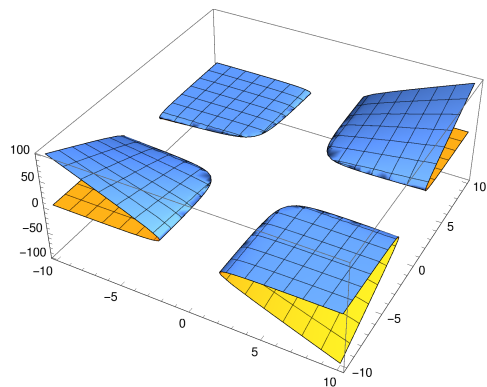
The six plots in Figure 4.1 show the behaviour of the preimage as we vary the point in the image.

#### *Case 1.*

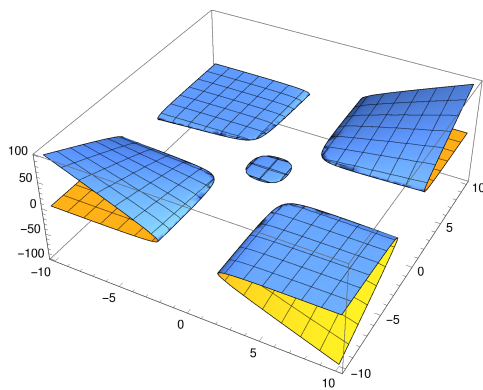
In this first case,  $\tau^{-1}(t)$  is made of representations which are holonomies of complete



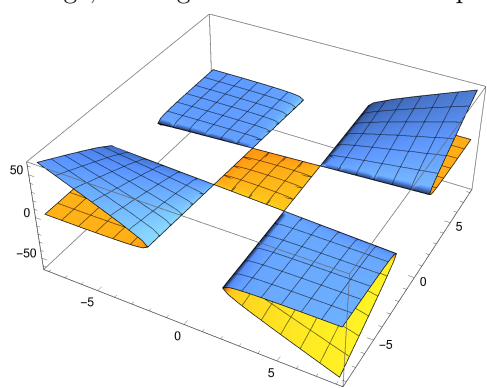
(a) Case 1:  $t=-5$



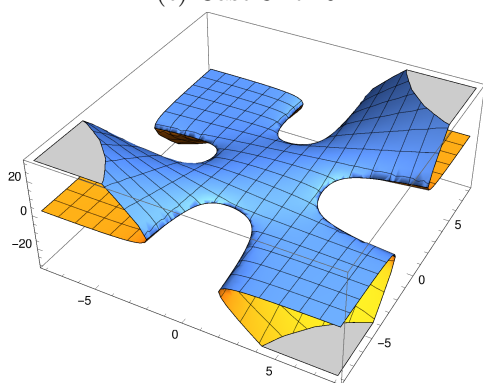
(b) Case 2:  $t=-2$  (the origin belongs to the preimage, although it is not visible in the plot)



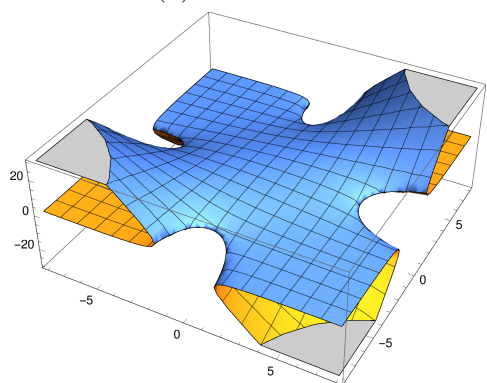
(c) Case 3:  $t=0$



(d) Case 4:  $t=2$



(e) Case 5:  $t=8$



(f) Case 6:  $t=25$

Figure 4.1: The different topologies of  $\tau^{-1}(t)$

hyperbolic structures on one-holed tori ([Gol2], §3.2). More precisely, they can be conjugated to Fuchsian representations which determine hyperbolic structures on one-holed tori. The commutator being loxodromic, the length of the boundary geodesic is strictly positive (the exact length can be computed by Equation 5.1), therefore there are no parabolic nor elliptic elements, and by Remark 4.2 the representation is Schottky. Hence, by Proposition 2.11 or by Theorem 3.8, we immediately get primitive stability and, consequently Bowditch conditions hold.

*Case 2.*

Here  $\tau^{-1}(t)$  is made of four copies of the Teichmüller space of a cusped torus and a singular point  $(0, 0, 0)$ . Similarly as above, points in the Teichmüller space correspond to equivalence classes of marked, complete, finite-area hyperbolic structures on a once-punctured torus. The trace of the commutator implies that the generator axes in the planar conjugate of the representation must cross. Again, by Theorem 3.8, we can conclude that the representation belongs to  $\mathcal{PS} \cap \mathcal{BQ}$ . Observe that here representations are not Schottky (being not purely loxodromic, thus Proposition 2.11 could not be applied). The singular point  $(0, 0, 0)$  is obviously outside of the  $\mathcal{BQ}$  set (and of  $\mathcal{PS}$ ).

*Case 3.*

When  $-2 < t < 2$ ,  $\tau^{-1}(t)$  has five connected components: one of these - call it  $C_t$  - is compact, whereas the others are not ([Gol2], §3.3). Representations in  $C_t$  are unitary, and unitary representations always have triples  $(x, y, z)$  such that  $-2 < x, y, z < 2$  ([Gol1], Section 5) so that they cannot belong to  $\mathcal{BQ}$ .

Alternatively, one can note that if the orbit of a representation lies in a compact, then the representation cannot be in  $\mathcal{BQ}$  by Remark 2.5. We also know that the action of  $\text{Out}(F_2)$  on  $C_t$  is ergodic, and we give below (Corollary 4.5) a different proof that this prevents the representations from being in  $\mathcal{BQ}$ . The representations in the other four connected components can be conjugated to get characters  $F_2 \rightarrow \mathbb{PSL}(2, \mathbb{R})$  which give singular structures on tori with one cone point. Applying Theorem 3.8, one gets primitive stability and belonging to  $\mathcal{BQ}$ . Note that the elliptic commutator can have finite or infinite order. When the order is finite, the image group in  $\mathbb{PSL}(2, \mathbb{C})$  is not free. When it is infinite, the group is not discrete. However, it is never Schottky.

*Case 4.*

When  $t = 2$ ,  $\tau^{-1}(t)$  consists of elementary representations, which cannot be in  $\mathcal{BQ}$ .

*Case 5.*

Here the preimage has only one connected component. Similarly to representations in  $C_t$  from Case 3., the action of  $Out(F_2)$  on  $\tau^{-1}(t)$  is ergodic for any  $t$  in this range ([Gol2], §5). According to Corollary 4.5, these representations must lie in the complement of  $\mathcal{BQ}$ . More concretely, one can notice that when  $t > 2$  and the representation is not conjugated to a Fuchsian one, by Theorem 5.2.1 in [Gol2] at least one entry in the triple  $(x, y, z)$  must belong to the interval  $[-2, 2]$ .

*Case 6.*

When  $t$  crosses the threshold  $t = 18$ , some wandering domains appear in  $\tau^{-1}(t)$ . These representations are discrete embeddings of  $F_2$  into  $\mathbb{P}SL(2, \mathbb{C})$  and give rise to complete hyperbolic structures on three-holed spheres ([Gol2], §5.1). Note that three holed spheres and one-holed tori have both fundamental group isomorphic to  $F_2$ . Let  $\Omega_0 = (-\infty, -2) \times (-\infty, -2) \times (-\infty, -2)$  and  $\Omega = Out(F_2) \cdot \Omega_0$ .  $\Omega \cap \tau^{-1}(t)$  is the subspace of  $\mathbb{R}^3$  corresponding to these structures.

For representations belonging to these specific domains, once again being Schottky assures that they belong to  $\mathcal{PS}$ , thus to  $\mathcal{BQ}$ .

$Out(F_2)$  acts ergodically on  $\tau^{-1}(t) \setminus \Omega$ , so the complement of  $\Omega$  in  $\tau^{-1}(t)$  is made of representations which do not belong to  $\mathcal{BQ}$  by Corollary 4.5 or by the same observation made in Case 5.

$\Omega \subseteq \mathbb{R}^3$  is the Fricke space of three-holed spheres, that is the set of points which correspond to holonomies of hyperbolic structures on three-holed spheres. As illustrated in Figure 4.1, Case 5. and Case 6. share the same preimage topological structures. Nevertheless, as the value of  $t$  increases, the preimage surface expands in  $\mathbb{R}^3$  and, from  $t = 18$  on, it starts intersecting the Fricke space. To see better how this happens, in Figure 4.2 we plotted one component of the Fricke space (namely  $\Omega_0$ ) and the preimage of  $\tau$  for a value of  $t$  in Case 5 (left), and one in Case 6 (right). The picture on the right shows a portion of the space where the surface intersects the Fricke space. Note that  $\Omega_0$  is the “parallelepiped” at the left bottom corner of both plots.

A well-known conjecture proposed by Bowditch says that if we restrict to representations with commutator trace equal to  $-2$ , then the primitive stable representations coincide with characters satisfying the  $BQ$  conditions. Another conjecture claims that the whole set  $\mathcal{PS}$  equals  $\mathcal{BQ}$ . Neither of these conjectures, as far as the author knows, has been proved or confuted so far. The result shown in this chapter, nevertheless, can be rephrased in the following theorem:

**Theorem 4.3.** *In the space of real representations, primitive stability holds if and*

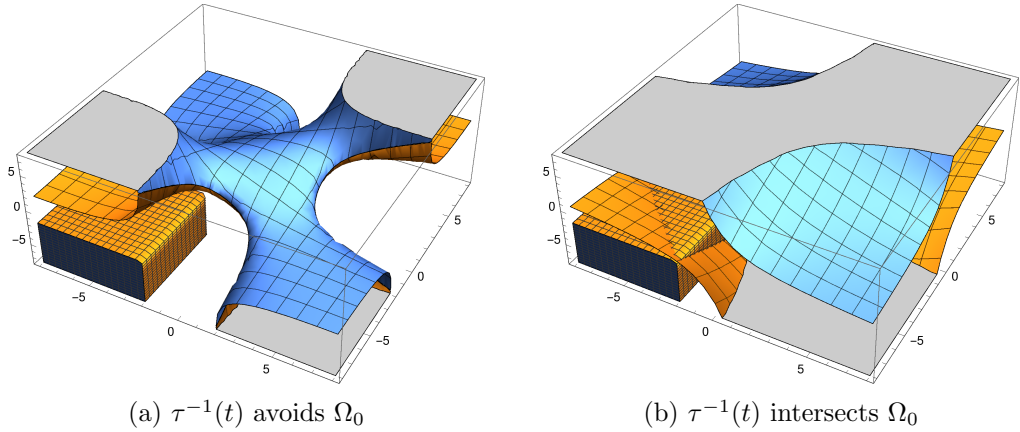


Figure 4.2:  $\tau^{-1}$  and  $\Omega_0$ .

only if Bowditch conditions hold.

### 4.3 Ergodicity and BQ conditions

We have seen above that, when dealing with the different cases for real representations, it turns out that all the components that do not lie in  $\mathcal{BQ}$  are characterised by an ergodic action of the group  $\Gamma$ . Although we did not need this information, before noticing that there was a shorter way to prove our claim, we tried to show that if  $\Gamma$  acts ergodically on a component, then no element of the component can belong to  $\mathcal{BQ}$ . For completeness, we report here the argument that one could alternatively use, hoping that it could be useful in some more general context, when one does not know any direct way of showing belonging to the complement of  $\mathcal{BQ}$ , but does know that the action of  $\Gamma$  is ergodic on some connected component.

In what follows we prove that the ergodicity of the action of the mapping class group on the character variety implies that for any open neighbourhood, almost every point in the neighbourhood comes back to it infinitely many times.

We say that a group  $\Gamma$  acts ergodically on a measure space  $(X, \mathcal{A}, \mu)$  if whenever a subset is invariant under the group action, then it has measure zero or its complement has. The measure  $\mu$  is said to be non-atomic if for any measurable set  $A \in \mathcal{A}$  with  $\mu(A) > 0$  it is possible to find a proper measurable subset  $S \subsetneq A$ , where  $0 < \mu(S) < \mu(A)$ . In other words,  $\mu$  is non-atomic if it does not admit atoms.

**Proposition 4.4.** *Let  $(X, \mathcal{A}, \mu)$  be a measure space where  $X$  is a topological space,  $\mathcal{A}$  is the  $\sigma$ -algebra of measurable sets in  $X$  and  $\mu$  is a non-atomic measure. Let  $\Gamma$  be a group acting on  $X$ , ergodically. Denote by  $\Gamma^* = \Gamma \setminus \{e\}$ . Then for any open set  $A \in \mathcal{A}$ , for  $\mu$ -almost every  $x \in A$ , the orbit  $\Gamma^* \cdot x$  intersects  $A$  at least once.*

*Proof.* Fix any finite, positive measure set  $A \in \mathcal{A}$  and define the set

$$E_A = \{x \in A \mid \Gamma^* \cdot x \cap A = \emptyset\}.$$

We want to show that  $\mu(E_A) = 0$ . First of all, we show that  $E_A$  must have either measure 0 or full measure in  $A$ . Obviously  $0 \leq \mu(E_A) \leq \mu(A)$ . Suppose that  $0 < \mu(E_A) < \mu(A)$  (note that, in particular, this implies that  $\mu(A \setminus E_A) > 0$ ). Then, by ergodicity, we have that  $\Gamma \cdot E_A$  has to cover the whole space  $X$ , up to nullsets. On the other hand, by definition of  $E_A$ ,  $\Gamma^* E_A \cap A = \emptyset$ , hence  $(A \setminus E_A) \cap (\Gamma \cdot E_A) = \emptyset$ , thus  $\mu(A \setminus E_A) = 0$ , giving a contradiction. Therefore  $\mu(E_A) = 0$  or  $\mu(E_A) = \mu(A)$ . We show that the latter case cannot occur. Suppose it does occur. Then, up to nullsets,  $E_A = A$ . Pick a proper measurable subset  $S \subsetneq E_A$ , with  $0 < \mu(S) < \mu(A) = \mu(E_A)$ . For the same argument as above,  $(\Gamma^* \cdot S) \cap A$  must have measure zero. Thus  $A \setminus S$  must have measure zero, contradicting the hypotheses. We conclude that  $\mu(E_A) = 0$ , as desired.  $\square$

**Corollary 4.5.** *With the same assumptions as above, if, in addition, the space  $X$  is Lindelöf (i.e. any open cover admits a countable subcover), then for any open set  $A \in \mathcal{A}$ , for  $\mu$ -almost every  $x \in A$ , the orbit  $\Gamma \cdot x$  intersects  $A$  infinitely many times.*

*Proof.* We start by defining the set

$$F_A = \{x \in A \mid |\Gamma \cdot x \cap A| < \infty\}.$$

Clearly we want to show that  $\mu(F_A) = 0$ . If  $x$  is a point in  $F_A$ , then we can find a neighbourhood  $S_x \ni x$ ,  $S_x \subseteq A$ , such that  $\Gamma^* x \cap S_x = \emptyset$ , that is  $x \in E_{S_x}$ . By the proposition above  $\mu(E_{S_x}) = 0$ . Now, write  $F_A$  as

$$F_A = \left( \bigcup_{x \in F_A} S_x \right) \cap F_A$$

Since  $X$  is Lindelöf, we can extract a countable family of  $S_x$ , say  $S_{x_1}, S_{x_2}, \dots$  such that

$$F_A \subseteq \bigcup_{i \in \mathbb{N}} E_{S_{x_i}}.$$

Consequently,  $\mu(F_A) < \sum_{i \in \mathbb{N}} \mu(E_{S_{x_i}}) = 0$  and we are through.  $\square$

**Remark 4.6.** *Corollary 4.5 shows that a generic representation in a subset of  $\mathbb{R}^3$  where  $\text{Out}(F_2)$  acts ergodically is almost-surely not in  $\mathcal{BQ}$ . Nevertheless,  $\mathcal{BQ}$  is an open subset of the character variety (see [TWZ]), therefore the ergodicity of the*

*action guarantees that each representation in a subset where the action is ergodic is actually not in  $\mathcal{BQ}$ .*



## Chapter 5

# The diagonal slice $\Delta$

In this chapter we introduce the diagonal slice  $\Delta$  of the character variety, that will be our main object of study for the rest of this paper. This subset, of complex dimension one, is rich of symmetries that can be exploited. Also, the low dimension allows us to comfortably plot pictures in the plane. We apply here some of the results obtained in the previous chapter regarding real representations. Points in the diagonal slice that do not lie on the real line will be treated in the next chapter, and by different techniques.

### 5.1 The slice and its symmetries

The diagonal slice, denoted by  $\Delta$  is defined as:

$$\Delta = \{[\rho] \in \chi(F_2, \mathbb{P}SL(2, \mathbb{C})) \mid \text{tr} \rho(a) = \text{tr} \rho(b) = \text{tr} \rho(ab)\}.$$

In short, we are only considering the classes of representations such that both the generators of  $F_2$  and their product are sent to matrices with the same complex traces.

Recall that the trace of a loxodromic element has an important geometric interpretation: if  $S \in \mathbb{P}SL(2, \mathbb{C})$  is a loxodromic isometry, let  $ax(S)$  denote its axis, which is by definition the geodesic line having the two boundary fixed points of  $S$  as endpoints. Then there exists a complex number  $l(S)$ , called the translation length of  $S$ , such that any point  $p \in ax(S)$  is moved at distance  $\Re l(S)$  and such that the normal to the axis at the point  $p$  is twisted by  $\Im l(S)$ . Note that  $l(S)$  is actually well defined only in  $\mathbb{C}/2\pi i\mathbb{Z}$ . The following formula relates translation lengths and

traces of loxodromic elements:

$$l(S) = 2 \cosh^{-1} \left( \frac{\text{tr} S}{2} \right). \quad (5.1)$$

The formula above gives us a geometrical interpretation of the diagonal slice: in the special case of a discrete, free representation, for instance, the handlebody  $\mathcal{H} = \mathbb{H}^3 / \rho(F_2)$  has a special 3-fold symmetry, due to the fact that the “three loops around the holes” have the same length and geometrical features. In [STY], the authors produced a very clear picture of the diagonal slice, that we are showing here in Figure 5.1. The picture represents a portion of the complex plane, where a point of complex coordinate  $x$  is associated to the representation defined by the triple  $(x, x, x)$ . The horizontal line which runs across the picture is the real line, corresponding to real parameter triples. The set  $\mathcal{BQ}$  is the complement of the black spot in the middle of the picture, and the complement of the ellipsoidal blue area is  $\mathcal{DF} \cap \Delta$ . Although we have good reasons to believe that the image is quite accurate in showing the set  $\mathcal{BQ}$ , there is no theoretical proof to confirm the validity of the picture. More precisely, the algorithm that produced the image proceeds by checking only finitely many points, so the impression of continuity is an illusion. Moreover, for each point the algorithm performs a huge amount of computer operations to determine if the point belongs to  $\mathcal{BQ}$ . If certain conditions are verified during this process, then the point must belong to  $\mathcal{BQ}$  with absolute certainty. If the conditions are not yet satisfied after many attempts, the algorithm can only “guess” that the point does not belong to  $\mathcal{BQ}$  and assigns it to its complement. Hence, we know that the  $\mathcal{BQ}$  set could actually be bigger than what is shown in the picture.

In [STY], the authors exhibit one possible choice of generator matrices for a conjugacy class of representations in the slice:

$$\rho_z: F_2 \rightarrow \mathbb{P}SL(2, \mathbb{C}),$$

$$\rho_z(a) = \begin{pmatrix} \frac{9}{4z^2} + \frac{1}{4} & -\frac{z^2}{4} + \frac{3}{4} \\ \frac{3}{4z^2} - \frac{1}{4} & \frac{z^2}{4} + \frac{1}{4} \end{pmatrix}, \quad \rho_z(b) = \begin{pmatrix} \frac{z^2}{4} + \frac{1}{4} & -\frac{z^2}{4} + \frac{3}{4} \\ \frac{3}{4z^2} - \frac{1}{4} & \frac{9}{4z^2} + \frac{1}{4} \end{pmatrix} \quad (5.2)$$

The parameter  $z$  in the formulae above and the parameter  $x$  corresponding to the “coordinate” in the complex plane shown in Figure 5.1 are related by the following equation:

$$\text{tr} \rho_z(a) = \text{tr} \rho_z(b) = \text{tr} \rho_z(ab) = \frac{9}{4z^2} + \frac{z^2}{4} + \frac{1}{2} = x. \quad (5.3)$$

**Remark 5.1.** *By Equation 5.3, it immediately follows that when  $x$  is real and*

$|x| \geq 2$ , then also  $z^2$  is real, and consequently the whole image of the representative character given by (5.2) sits inside  $\mathbb{PSL}(2, \mathbb{R})$ . In what follows, we shall say that “a representation is Fuchsian” as a short form for “the conjugacy class corresponding to a triple contains one Fuchsian representative”.

The remaining part of this paper is devoted to showing the following main theorem:

**Theorem 5.2.** *Let  $\mathcal{DF}$  be the subset of discrete and free representations. Then  $\mathcal{DF} \cap \Delta$  is strictly contained in  $\mathcal{PS} \cap \Delta$ . In particular, there exists a whole neighbourhood of representations in the slice which are primitive stable but do not belong to  $\mathcal{DF}$ .*

Here is an outline of the proof:

Consider  $\Delta$  as the ambient space. Points in the interior of  $\mathcal{DF}$  are Schottky, hence they belong to  $\mathcal{PS}$  by Proposition 2.11. We need to prove that every point (with a few exceptions) in  $\partial\mathcal{DF}$  belongs to  $\mathcal{PS}$ . Since primitive stability is an open condition ([Min], Lemma 3.2), this would immediately imply the statement of the theorem. We shall prove that the boundary of  $\mathcal{DF}$  is contained in  $\mathcal{PS}$  in the next chapter.

## 5.2 Real representations in $\Delta$

Below, we consider real representations belonging to the diagonal slice. As one can see from the picture, when restricting to real values for  $x$ , we can observe different behaviours for the corresponding representations. As usual, the main piece of information comes from the commutator trace  $\tau$ , that relates to the real parameter  $x$  through

$$\tau = \tau(x) = 3x^2 - x^3 - 2.$$

Figure 5.2 shows a plot of this polynomial function in the relevant range. The vertical and horizontal grey lines in the picture show when the function passes through important thresholds, as discussed in Chapter 4. Note that:

$$\tau(-2) = 18, \tau(-1) = 2, \tau(0) = -2, \tau(2) = 2, \tau(3) = -2.$$

Outside the plotted range, the function does not cross the horizontal strip  $\mathbb{R} \times [-2, 2]$ . We organise the analysis of the real line in the diagonal slice in a short sequence of lemmas.

**Lemma 5.3.** *Representations corresponding to  $[3, \infty)$  are primitive stable.*

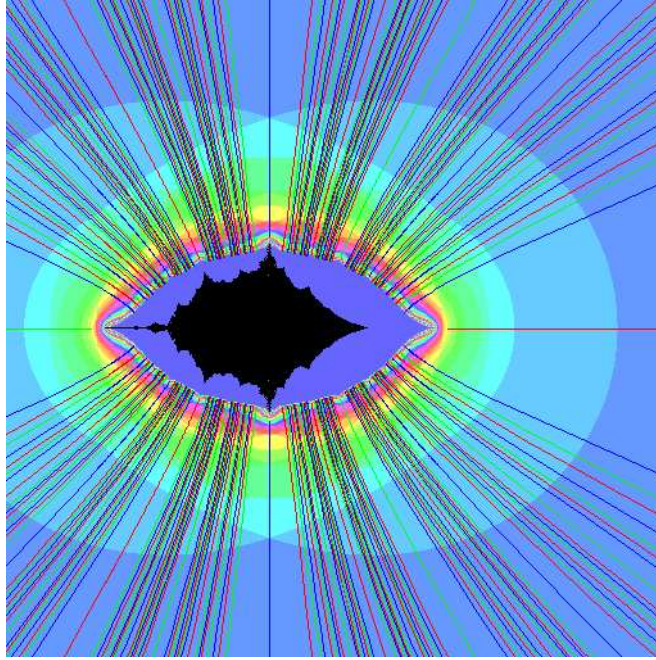


Figure 5.1: A picture of the slice  $\Delta$ .

*Proof.* This is basically Case 1 and Case 2 from Chapter 4. The representations are Fuchsian and correspond to holonomies of one holed/punctured tori.  $\square$

**Lemma 5.4.** *Representations corresponding to  $(2, 3)$  are primitive stable.*

*Proof.* This follows from Case 3 in Chapter 4, since we know that in the diagonal slice representations with real parameters with absolute value greater than or equal to 2 can always be conjugated to Fuchsian ones, hence they are not unitary. Note that the interval  $(2, 3)$  does not belong to  $\mathcal{DF}$ , as the commutator element is always elliptic.  $\square$

When  $x$  equals 2 or  $-1$ , the commutator trace equals 2. This is the case of elementary representations, which do not belong to  $\mathcal{BQ}$ .

For points in  $(-2, 2)$ , the generators are elliptic, and for  $x = 2$  they are parabolic, thus the representations cannot satisfy BQ conditions. Therefore:

**Lemma 5.5.** *Representations corresponding to  $[-2, 2]$  do not satisfy Bowditch conditions.*

The only case which is left is the following:

**Lemma 5.6.** *Representations corresponding to  $(-\infty, -2)$  are primitive stable.*

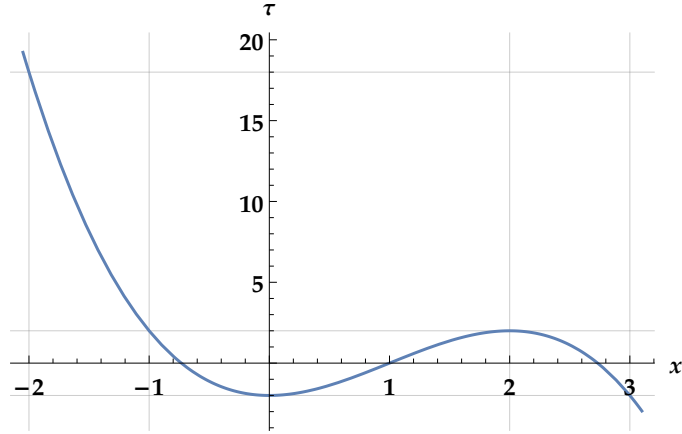


Figure 5.2: A plot of the commutator trace function for real values of  $x$ .

*Proof.* We are in Case 6 from Chapter 4. To see that we are actually in the subset  $\Omega$  corresponding to holonomies of hyperbolic structures on three holed/punctured spheres, it is enough to show that they belong to  $\mathcal{DF}$ . It turns out that the semiline  $(-\infty, -2]$  is the pleating ray for the multicurve whose connected components are the loops represented by  $a$ ,  $b$  and  $ab$ , and that the endpoint  $-2$  corresponds to pinching the three boundary components of the holed sphere to cusps. A proof of this fact can be found in [STY]. The theory of pleating rays is explained in the following chapter.  $\square$

The lemmas above characterise all representations on the real line in the diagonal slice. Observe that, as we already knew from Chapter 4, primitive stability and Bowditch conditions are equivalent for real representations.

## Chapter 6

# Pleating rays and boundaries of $\mathcal{PS} \cap \Delta$

The aim of this last chapter is to prove that, in the diagonal slice, every point in  $\partial\mathcal{DF}$  belongs to  $\mathcal{PS}$  (with the exception of three special ones). Thanks to some deep results that follow from the ending lamination theorem, it is well known that points in the boundary of  $\mathcal{DF}$  can be of two different types: geometrically finite cusp groups and geometrically infinite groups with irrational ending laminations. The former type of groups arises when approaching the boundary of  $\mathcal{DF}$  by starting from its interior, made of Schottky representations, and continuously deforming the character by selecting one closed simple, essential multicurve on the boundary of the quotient handlebody, and pinching it to a cusp. Groups arising in this fashion are examined in Section 6.2. The remaining points on the frontier correspond to geometrically infinite groups whose quotient is a non compact 3-manifold homeomorphic to  $\mathcal{H} \times [0, 1)$  and whose ending lamination is irrational (i.e. not a union of simple closed curves). This case is exposed in Section 6.3 below.

### 6.1 A bit of pleating ray theory

The theory of deformation of Kleinian groups is well understood and described in a variety of papers and books. Among the different techniques that one could use, we are interested in the theory of pleating rays. As we know, Schottky representations are always primitive stable. Pleating ray theory equips us with a tool to determine certain semilines in  $\mathcal{DF}$  (called *pleating rays*, indeed) ending at  $\partial\mathcal{DF}$ , corresponding to cusps in the boundary. In what follows, we shall use some of the many known facts about pleating rays, introduced by Keen and Series in [KS2], and vastly exposed

and developed in [KS1], [KS3] and [KS4]. The main idea in our case of interest is the following: when the representation is Schottky, the quotient manifold is a genus 2 handlebody, whose conformal boundary is homeomorphic to a closed surface of genus 2. The image of  $F_2$  inside  $\mathbb{PSL}(2, \mathbb{C})$  acts on the hyperbolic space  $\mathbb{H}^3$  and its conformal boundary, the Riemann sphere. The action partitions the Riemann sphere into two subsets: the limit set, where the action of the group is ergodic, and its complement, the regular set, where the action is properly discontinuous. The quotient of the convex hull of the limit set is called the convex core, and it is a pleated surface in the quotient 3-manifold, homeomorphic to the boundary of the handlebody, that is a closed surface of genus 2. This pleated surface is bent along certain geodesics, which can be simple multicurves or actual laminations. Keen and Series showed that if a pleated surface is bent along a simple curve  $\gamma$ , then  $\gamma$  is the projection to the quotient of the axis of a loxodromic element whose trace is real. In [KS4], the authors develop a powerful method to investigate the discrete locus of representations in the Riley slice, that is representations of the free group of rank 2 when both generators are parabolic. The same techniques turn out to be very powerful in the diagonal slice too.

Assume that we are given a Schottky representation  $\rho: F_2 \rightarrow \mathbb{PSL}(2, \mathbb{C})$ , in  $\Delta$ . Let us look at the quotient:  $\mathbb{H}^3/G$  is isomorphic to a hyperbolic handlebody of genus 2 and its boundary  $\partial\mathcal{H}$  is a hyperbolic closed surface of genus 2. Since  $\rho \in \Delta$ , the handlebody, hence the surface, has a 3-fold rotational symmetry  $\Omega$ , which is the isometry that rotates the handlebody by  $2\pi/3$  radians about the virtual axes going through the “poles” as shown in Figure 6.1. It is easy to see that the quotient  $\partial\mathcal{H}/\Omega$  is geometrically a sphere with four cone points around each of which the angle is  $2\pi/3$ . Similarly, the quotient of the handlebody is a 3-dimensional orbifold containing two singular axes joining pairs of cone points in the boundary, around which the angle is  $2\pi/3$ . Following [STY], we call the 2-dimensional orbifold the *large coned sphere*, and we denote it by  $\partial\mathcal{S}$ . We call the whole 3-dimensional orbifold the *large coned ball*, and we denote it by  $\mathcal{S}$ . Much of the geometrical information about  $\partial\mathcal{H}$  is encoded in this quotient. In particular we shall be interested in parametrising all simple closed curves on  $\partial\mathcal{H}$  which are invariant under  $\Omega$ . We shall see below how this can be comfortably achieved by looking at the quotient.

Note that the holonomy group of the handlebody and that of the large coned sphere are commensurable. Hence, they must be both discrete or not discrete at the same time. The big advantage of working with the large coned sphere instead

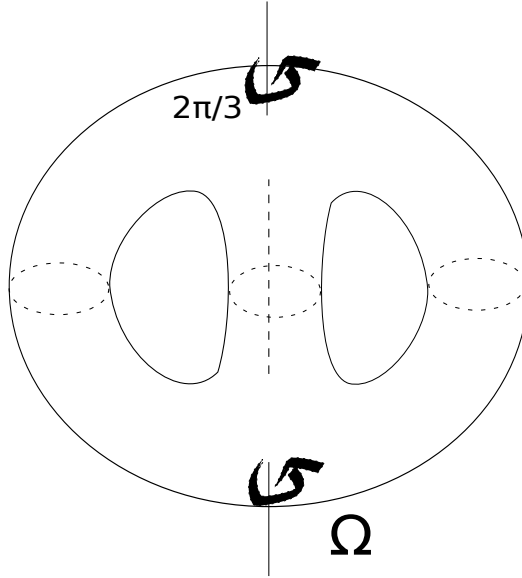


Figure 6.1: The rotation  $\Omega$  around the line passing through the poles. Note that it has four fixed points on the surface and two fixed arcs in the handlebody.

of the handlebody is that simple closed curves are more easily understood on the former. In [STY], the authors prove the following result:

**Proposition 6.1** ([STY], Proposition 4.8). *Essential nonperipheral unoriented simple curves on  $\partial\mathcal{S}$  are, up to homotopy, in bijective correspondence with  $\mathbb{Q} \cup \infty$ . If  $\gamma_{p/q}$  (for  $p$  and  $q$  relatively prime) denotes the curve corresponding to  $p/q$ , moreover, then the unoriented curves  $\gamma_{p/q}, \gamma_{p'/q'}$  are homotopic in  $\mathcal{S}$  if and only if  $p'/q' = \pm p/q + 2k$ ,  $k \in \mathbb{Z}$ .*

The meaning of the integer numbers  $p$  and  $q$  in the enumeration explained in the proposition above has to do with a special way of drawing curves on the large coned sphere. The large coned sphere admits a Klein group of symmetry, where the three involutions forming the group consist in  $\pi$  rotations about the three “orthogonal axes” passing through its centre. Imagine splitting  $\partial\mathcal{S}$  along its unique meridian, as illustrated in Figure 6.2. Each simple closed curve which is not the meridian, can be drawn in the following way, up to homotopy: start by drawing  $q$  non-overlapping parallel arcs on each hemisphere, each of which has both endpoints on the boundary given by cutting along the meridian. Now glue each left strand with the right strand which is  $p$  positions “higher” than the left one, i.e. twist the curve by  $p$  steps to the left. This procedure is carefully explained in [STY]. We assign the parameter  $p/q$  to the curve obtained this way. Since the meridian is the



only simple closed curve which cannot be obtained in this fashion, we arbitrarily assign to it the parameter  $1/0$ . A similar way of drawing curves on surfaces and

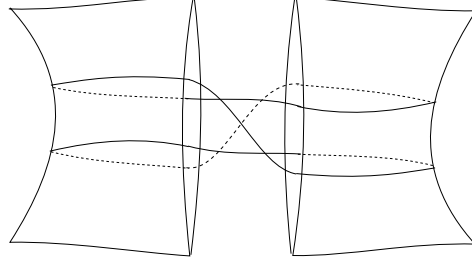


Figure 6.2: The large coned sphere, split along the meridian into two hemispheres, and the simple closed curve corresponding to  $1/2$ .

enumerating them by integer parameters will be very useful later on, when we will be dealing with  $\Omega$ -symmetric curves on the boundary of the handlebody. In [STY] the authors proceed by working out each of these steps:

- enumerate all simple closed curves on the large coned sphere,
- compute the trace of the corresponding loxodromic matrices,
- determine the real locus for the trace of each matrix,
- plot the “pleating” rays in the diagonal slice.

The results obtained are surprisingly neat. In particular, the authors prove that for each simple closed curve, the intersection of the real locus of the associated matrix and  $\mathcal{DF}$  is never empty and it is formed by either one or two connected components (a connected pleating ray occurs only in two special cases, all the remaining being disconnected). Moreover they compute the asymptotic direction of the rays, and show that the pleating rays form a dense subset of the discrete locus of the character variety.

In Figure 5.1, pleating rays are plotted in the diagonal slice. As we move along a pleating ray towards the boundary in the “centre” of the picture, it is important to understand what is happening from a geometrical point of view. The endpoint of the pleating ray associated to a curve on  $\partial\mathcal{H}$  which is the projection of the axes of a loxodromic isometry with matrix  $S$ , corresponds to the equation  $tr(S) = \pm 2$  or, in other words, we stop plotting the ray as soon as the examined element is

pinched to a parabolic, and in the quotient we see a cusp. The reason why we stop here is that, after this threshold, the examined element would become elliptic, and the representation would be either not free (when the elliptic element has finite order) or not discrete (when it has infinite order). As we approach this endpoint, we are pinching a curve to a cusp, without generating any other accidental parabolic elements in the image group. Points in the boundary of  $\mathcal{DF}$  which originate in this fashion are called cusps, and they are dense in  $\partial\mathcal{DF}$ .

In the next section, we show that almost all the geodesics on  $\partial\mathcal{H}$  which are invariant under the rotation  $\Omega$  correspond to blocking words, and by a result of Minsky we derive that the endpoints of the associated pleating rays in the boundary of  $\mathcal{DF}$  correspond to primitive stable characters. In the last section of the chapter, we complete the picture by dealing with the remaining points on the boundary of  $\mathcal{DF}$ .

## 6.2 Rational case: lifted curves are blocking

The main goal of this section is to prove that simple, essential, closed curves on  $\partial\mathcal{H}$  that are invariant under  $\Omega$  correspond to blocking words in  $F_2$ .

We allow curves to be not connected, that is to be multicurves. Whichever adjective we use for a multicurve (e.g. closed, or simple) applies separately to each of its connected components. It is clear that a curve  $\gamma$  on  $\partial\mathcal{H}$  such that  $\Omega(\gamma) = \gamma$  projects down to a curve on  $\partial\mathcal{S}$ . Vice versa, if we start with a curve on  $\partial\mathcal{S}$  and we lift it to  $\partial\mathcal{H}$ , then we get a curve with either one or three connected components, and invariant under  $\Omega$ . Hence, since we want to investigate pleating rays of essential, nonperipheral simple closed curves on the large coned sphere, equivalently we can consider their lifts on  $\partial\mathcal{H}$ . As we have already observed, curves on  $\partial\mathcal{S}$  are bijectively enumerated by the extended rational numbers  $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ . Moreover, the pleating rays corresponding to these curves are dense in the discrete and free locus of the diagonal slice. We call a curve on  $\partial\mathcal{H}$ , which is simple, closed, essential and invariant under  $\Omega$ , a  $\Omega$ -symmetric curve. The main idea of the section is the following: we enumerate  $\Omega$ -symmetric curves by rational numbers and prove that all but finitely many of these curves correspond to blocking elements in the free group. The key proposition we shall prove is the following.

**Proposition 6.2.** *Every  $\Omega$ -symmetric curve, with the exception of those corresponding to  $0/1$  and  $1/2$  in the rational enumeration scheme, is blocking.*

Then we combine this with a theorem of Minsky to get our main result of the section:

**Theorem 6.3.** *The points in  $\partial\mathcal{DF} \cap \Delta$  reached by pinching  $\Omega$ -symmetric curves (apart from  $\gamma_{0/1}$  and  $\gamma_{1/2}$ ) correspond to primitive stable characters.*

Since primitive stability is an open condition, we immediately get the following corollary:

**Corollary 6.4.** *The set  $\mathcal{PS} \cap \Delta$  is strictly bigger than the discrete, free locus in  $\Delta$ .*

To better encode algebraically the geometrical symmetry of the handlebody, we will think of  $F_2$  with the following redundant presentation:

$$F_2 = \langle a, b, c | abc \rangle.$$

Geometrically,  $a$ ,  $b$ , and  $c$  describe the three oriented loops around the “holes” of the handlebody. Algebraically, this means that  $\Omega(a) = b$ ,  $\Omega(b) = c$ ,  $\Omega(c) = a$ .

For the enumeration scheme, if  $\gamma$  is an  $\Omega$ -symmetric curve, one can project it down to the orbifold  $\partial\mathcal{S}$ , getting an essential, non peripheral, simple closed curve  $\gamma_{p/q}$ ; we use the same parameter to enumerate the  $\Omega$ -symmetric lifts in  $\partial\mathcal{H}$ . If  $p \equiv q \equiv 1 \pmod{2}$  then the lifted curve has only one component and the corresponding word  $W_{p/q}$  in  $F_2$  has length equal to  $3q$ . If not, the lifted curve has three connected components and the corresponding words have each length equal to  $q$  and are cyclically permuted by  $\Omega$ .

First of all, note that every  $\Omega$ -symmetric curve can be described in the following way: consider the handlebody as split into two halves and arrange strands in the two “hemispheres” as in Figure 6.3. Then join the top strands with the bottom strands by “twisting”  $p$  steps to the right. Note that the  $\Omega$ -invariance implies that the number of strands on each “connection” is the same and that the twist is fixed too. In other words, we can describe such a curve by giving the number of strands at each joint (note that this has to be an even number) and the amount of twisting. If we denote by  $\hat{\gamma}_{p/q}$  the  $\Omega$ -symmetric curve obtained as described above, it is crucial to observe that  $\hat{\gamma}_{p/q}$  is the lift of the curve  $\gamma_{p/q}$  on the large coned sphere, so that the enumeration scheme for the large coned sphere lifts perfectly to a rational enumeration scheme for  $\Omega$ -symmetric curves on the boundary of the handlebody. Figure 6.3 shows the case where  $2q = 6$  and  $p = 2$ , where  $2q$  is the number of strands passing through each of the joints.

In order to prove Proposition 6.2, let us subdivide the upper and lower strand endpoints into 12 blocks as in Figure 6.5, and assign to each of the  $3 \cdot (2q)$  horizontal positions a progressive number, from 1 to  $6q$ . Here and in what follows, a *block* is a

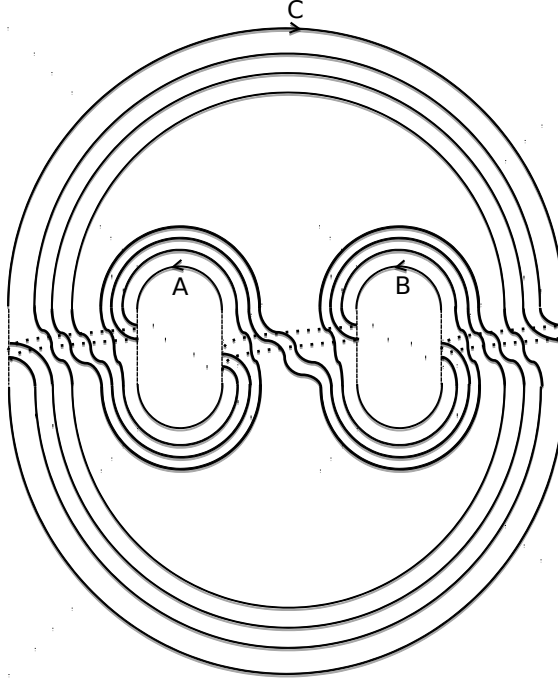


Figure 6.3: A generic  $\Omega$ -invariant simple closed curve on the boundary of the handlebody.

set of endpoints which lie on the boundary of one of the two hemispheres. Each of the six boundary components (three per hemisphere) contains two blocks. If we consider a natural circular ordering on the set of endpoints on each boundary components, each block contains  $q$  consecutive endpoints. The blocks are labelled with letters  $A, \dots, N$  as in Figure 6.5. A *position* is the label of a specific endpoint. Hence, there are  $12 \cdot q$  positions, evenly distributed across 6 connected components, or 12 blocks. If we draw the surface flat, as in Figure 6.5, positions on each hemisphere (thus on each row) are labelled from left to right with natural numbers, so that we assign to each number two different endpoints, one on the top row and one on the bottom one. To distinguish among them, we add to the label a subscript  $t$  or  $b$  according to whether the endpoint is on the top or bottom row, respectively. For instance  $7_t$  will correspond to the upper endpoint of the seventh strand, counting from left to right, and  $(6q)_b$  to the last endpoint in the lower hemisphere. Obviously, when the two hemispheres get glued along the boundaries to form the original surface, positions with the same number and different subscripts get identified. This notation will be very handy later on.

By Remark 2.2, primitive words in  $F_2$  are always expressed (after cyclic reduction)

by balanced sequences in two letters. In particular, a cyclically reduced word that cannot occur as a subword of a balanced sequence is always primitive blocking.

An easier way to code and track curves of the type described above on the genus two surface is to write them in terms of arcs. We consider the three arcs on the handlebody as showed in Figure 6.4. A *full loop*, or just *loop*, is the path

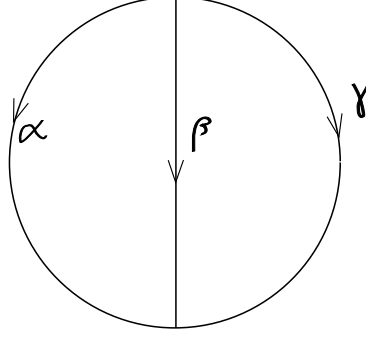


Figure 6.4: Arcs provide an easier way to track curves on  $\delta\mathcal{H}$ ,

determined by one of the three generators  $a$ ,  $b$  and  $c$ , or its inverse. In terms of paths on the surface, it corresponds to going once around one of the three holes of the surface, starting and ending at a fixed point in the upper hemisphere. In terms of blocks, it is determined by any sequence of blocks that can be traversed by one of these paths. For instance, the sequence

$$B \rightarrow H \rightarrow I \rightarrow C$$

describes the loop given by the generator  $A$ . In terms of positions, it is determined by any sequence of four positions which belong, in order, to a block in a sequence of four determining the loop as described above. The relations between these arcs and full loops are:

$$a = \alpha\beta^{-1}, \quad b = \beta\gamma^{-1}, \quad c = \gamma\alpha^{-1}.$$

Each loop being determined by two arcs, this notation allows for a finer control of the paths on the surface. If

$$w = w(a, b) = ab^{n_1} \dots ab^{n_p},$$

we can rewrite it in term of arcs as:

$$w = w(\alpha, \gamma, \beta) = \alpha\gamma^{-1} (\beta\gamma^{-1})^{n_1-1} \dots \alpha\gamma^{-1} (\beta\gamma^{-1})^{n_p-1}. \quad (6.1)$$

The two expressions  $\alpha\gamma^{-1}$  and  $\beta\gamma^{-1}$  correspond, respectively, to going once around the  $C^{-1}$  loop (outermost circle, anticlockwise) and once around the  $B$  loop (rightmost, clockwise). We want to show that the word  $w'$  associated to an  $\Omega$ -symmetric curve is primitive blocking by contradiction. Suppose it is not. Then it must be a subsequence of a balanced sequence. Suppose, without loss of generality, that the two letters forming the word are  $a$  and  $b$ , where  $a$  is the isolated letter, always occurring with exponent 1, and  $b$  is the non isolated one, occurring with positive exponents. Being balanced,  $w'$  can always be completed by adding letters to its left and right to look like  $w = ab^{n_1} \dots ab^{n_p}$ , where  $n_i \in \{M, M+1\}$  and  $M \geq 0$ . We choose  $w$  to be the minimal completion of  $w'$  to a word of this form.

Firstly, we show that the integer  $M$  cannot be greater than 2.

**Proposition 6.5.** *If  $w = ab^{n_1} \dots ab^{n_p}$ ,  $n_i \in \{M, M+1\}$ , is the minimal extension of a word which represents an  $\Omega$ -symmetric curve, then  $M \leq 2$*

In order to show this, we need:

**Lemma 6.6.** *Let us return to considering  $\Omega$ -symmetric curves on  $\partial\mathcal{H}$ , with the division of upper and lower strand endpoints into blocks as above. If we keep moving around the same loop, each time we come back to the starting block the curve progresses according to an arithmetic progression with step distance  $2p$  to the right or to the left.*

Before moving into a more formal argument, we start by giving an example. Imagine to start from  $1_t$ , and that the twist value is set to  $p = 1$ , with  $q = 10$  (i.e. each strand is twisted to the right by two steps, following it from top to bottom). If we track the word  $w = \dots c^{-1}c^{-1}c^{-1} \dots$ , the corresponding block sequence, starting from block  $A$ , will be

$$(A \rightarrow G \rightarrow N \rightarrow F) \rightarrow (A \rightarrow G \rightarrow N \rightarrow F) \rightarrow (A \rightarrow G \rightarrow N \rightarrow F).$$

At a more precise level, the exact position sequence, starting from 1 in block  $A$ , will be:

$$1_t \mapsto 2_b \mapsto 59_b \mapsto 58_t \mapsto 3_t \mapsto 4_b \mapsto 57_b \mapsto 56_t \mapsto 5_t \mapsto 6_b \mapsto 55_b \mapsto 54_t.$$

Therefore, in block  $A$  the position sequence is  $1, 3, 5, \dots$ , in block  $G$  it is  $2, 4, 6, \dots$ , in block  $N$  it is  $59, 57, 55, \dots$  and in block  $F$ , finally,  $58, 56, 54, \dots$ . Hence, since in a balanced sequence the isolated letter always has exponent equal to 1, by Lemma 6.7 we deduce that the exponents of the pattern  $\beta\gamma^{-1}$  must be either one or two.

Equivalently  $n_i \in \{2, 3\}$ ,  $\forall i$ . After these preliminary observations, we can proceed with a formal argument.

*Proof of Lemma 6.6.* We follow an  $\Omega$ -symmetric curve on the surface, starting from a point in the top hemisphere. The first four positions we traverse always occur in the order top, bottom, bottom, top and determine one generator letter (or its inverse). For instance, in the example above, the first four positions

$$1_t \mapsto 2_b \mapsto 59_b \mapsto 58_t$$

determine the letter  $C^{-1}$ . Note also that, although the sequence values between a top position and a bottom position depend on the parameter twist, two adjacent values in the sequence belonging to the same row must always appear paired, since they are joined by one of the fixed strands that we arranged on the surface. Given a starting position  $a = a_t$ ,  $a \in \{1, \dots, 6q\}$  in the top row of a block and the twist parameter  $p$ , we can write the position transformation laws in terms of  $p$  and  $a$ . In particular,  $a$  and  $p$  contain enough information to determine the whole sequence of positions that the (connected component of the) curve must pass through before closing up on itself (and therefore also the corresponding word). By writing the transformation formulae modulo  $q$ , we only care about the relative position in the block that the curve is traversing at each step, and lose track of which block we are looking at. Let us write down the formulae, labelling the strand endpoints by  $s_1, s_2, \dots$ , in the order we come across them as we track the curve: if we start in  $s_1 = a_t$  (whose relative position in the block is  $a \pmod{q}$ ), we go to the bottom row by moving down the surface and twisting by  $p$  (where a positive value of  $p$  gives a twist to the right, as we move from the top to the bottom row, and a negative value gives a left twist). The new relative position of  $s_2$  in the bottom row, modulo  $q$ , would then be  $a + p \pmod{q}$ . As we keep following the curve, we now travel from  $s_2$  in the bottom row to  $s_3$ , in the bottom row of a different block. Since the first bottom position of a block is linked by a strand to the last of the block to which it is linked, the second to the second last, and so on until the last that is linked to the first, to find  $s_3$  we reflect  $s_2$  in the centre line of the diagram of Figure 6.5, thus  $s_3$  has relative position  $-(a + p) \pmod{q}$ . We now go from the bottom row to  $s_4$  in the top one, and twist left. Therefore, the relative position of  $s_4$  is:  $-(a + p) - p = -(a + 2p) \pmod{q}$ . In the fourth and last step, we follow a strand in the top hemisphere. Similarly to what happened before, we reflect  $s_4$ , getting  $s_5$  with position  $-[-(a + 2p)] = a + 2p \pmod{q}$ . This shows that, regardless of the specific block sequence, an  $\Omega$ -symmetric curve will always describe an arithmetic

progression in terms of relative positions.  $\square$

Finally, here is the technical lemma we need to conclude the proof of Proposition 6.5.

**Lemma 6.7.** *Let  $a, \rho \in \mathbb{Z}$ ,  $\rho \neq 0$ . Consider the set  $M = \{a + \rho b \mid b \in \mathbb{Z}\}$ . Fix a natural length  $l > 0$  and split  $\mathbb{Z}$  into consecutive subsets  $B_i$ , each made of  $l$  consecutive numbers. That is,*

$$\mathbb{Z} = \coprod_{i \in \mathbb{Z}} B_i$$

*Then there exists a natural number  $k$  such that, for each  $i$ ,  $|B_i \cap M| = k$  or  $k + 1$ .*

*Proof.* First of all, if we denote by  $\max_i$  and  $\min_i$  the maximum and minimum elements of  $B_i$ , respectively, it is clear that  $l = \max_i - \min_i + 1$ . Suppose that  $B_i$  contains  $k + 1$  elements, for some  $k$ . This implies that there exists  $h \in \mathbb{Z}$  such that

$$\rho h, \rho(h + 1), \dots, \rho(h + k) \in B_i.$$

Then:

$$l = \max_i - \min_i + 1 \geq \rho(h + k) - \rho(h) + 1 = \rho(k) + 1$$

so that  $l \geq \rho(k) + 1$ . On the other hand, since only  $k + 1$  elements occur in the set, it must be that  $\max_i < \rho(h + k + 1)$  and  $\min_i > \rho(h - 1)$ . Putting things together,  $l < \rho(k + 2) + 1$  and, finally;

$$\rho(k) + 1 \leq l < \rho(k + 2) + 1.$$

Now, if there were another block containing  $k - 1$  elements from  $M$ , this would imply, by the same argument:

$$\rho(k - 2) + 1 \leq l < \rho(k) + 1,$$

giving the contradiction  $\rho(k) + 1 < \rho(k) + 1$ .  $\square$

Since in a balanced sequence one letter is always isolated, Lemma 6.6 and Lemma 6.7 suffice to prove the statement of Proposition 6.5: the twist parameter of each curve is fixed, determining an arithmetic progression on the set of positions modulo  $q$ . Since the exponent of one of letter is always one, the  $n_i - 1$  exponents in (6.1) are less than or equal than 2, hence  $n_i \leq 3$  for each  $i$ . Hence we know that every balanced sequence with exponents higher than 2 cannot occur as a lift from a



curve on the large coned sphere.

To rule out the remaining low exponent cases, where  $M \leq 2$ , we argue according to the following idea: the parameter  $p$  tells us, naively, the reluctance of the curve to abandon a pattern and start going through a different loop. A small value of  $p$  (relatively to the number of strands) will give rise to a curve that spirals many times around each loop, whereas a value of  $p$  which is nearly half of the number of strands will produce a curve that changes patterns often. We want to show that the pattern

$$\alpha\gamma^{-1}(\beta\gamma^{-1})\alpha\gamma^{-1}(\beta\gamma^{-1})^2 \quad (6.2)$$

already violates the curve structure. Every primitive word with low exponents must contain this pattern (provided it is long enough). Hence, this would show that there are no low exponent  $\Omega$ -symmetric curves which are not primitive blocking.

We start by convincing ourselves that  $p$  can always be chosen to be a positive number (that is, we twist strands towards the right) less than or equal to  $q$ . To see this, we observe first that twisting towards the right by a certain value  $p$  is the same as twisting towards the left by  $2q - p$ . Moreover, the unoriented (multi)curve obtained by twisting to the right is equal to the one obtained by twisting by the same amount of steps to the left. This allows us to restrict to  $p \in \{1, \dots, q\}$ .

The block sequence corresponding to pattern (6.2) is:

$$\begin{aligned} \dots \rightarrow A \rightarrow G \rightarrow N \rightarrow E \rightarrow D \rightarrow L \rightarrow M \rightarrow F \rightarrow \\ \rightarrow A \rightarrow G \rightarrow N \rightarrow E \rightarrow D \rightarrow L \rightarrow \\ \rightarrow M \rightarrow E \rightarrow D \rightarrow L \rightarrow M \rightarrow F \rightarrow A \rightarrow G \rightarrow \dots \end{aligned}$$

Moreover, the exact position in the sequence of blocks will be determined exactly as it was done in the discussion preceding the proof of Lemma 6.6. We now look for a contradiction in the equations gotten while going through the sequence of blocks. This was suggested by computer simulations: the script we used just tried to start from each position in block  $A$  and follow the sequences determined by all the possible twists (from 1 to  $q$ ). We noticed that, at most after 15 “steps”, the curve would land on a block which is not the correct one, according to the patterns we want to follow. To verify that this is true, we need to write the exact positions that occur while moving along the pattern, in terms of  $p$ ,  $s$  (the starting position in block  $G$ ) and  $q$ . The integer number of each position must belong to a specific block; this gives two inequalities for each position in the sequence. After fifteen steps, we have therefore a system of thirty inequalities. Here is the explicit computation. We assume to start

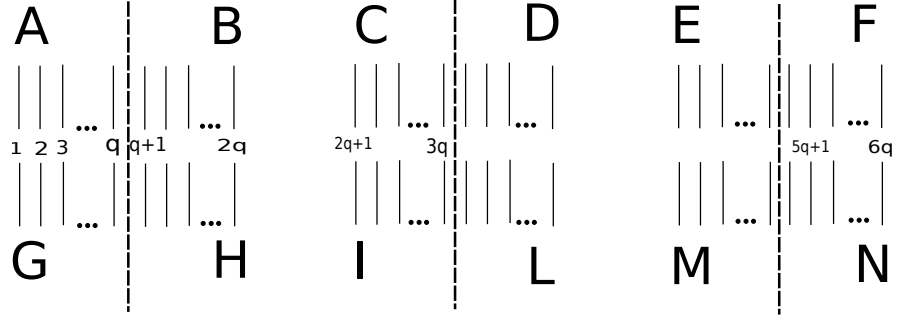


Figure 6.5: Position and block labelling.

in position  $s_b$  from block  $G$ , and keep following. We add a superscript for the block too.

$$\begin{aligned}
& s_b^G \mapsto (6q - s + 1)_b^N \mapsto (6q - s + 1 - p)_t^E \mapsto (2q + s + p)_t^D \mapsto \\
& \mapsto (2q + s + 2p)_b^L \mapsto (6q - s - 2p + 1)_b^M \mapsto (8q + 1 - s - 3p)_t^F \mapsto \\
& \mapsto (-2q + s + 3p)_t^A \mapsto (s - 2q + 4p)_b^G \mapsto (8q - s - 4p + 1)_b^N \mapsto \\
& \mapsto (8q - s + 1 - 5p)_t^E \mapsto (5p + s)_t^D \mapsto (6p + s)_b^L \mapsto (8q + 1 - 6p - s)_b^M \mapsto \\
& \mapsto (8q + 1 - 7p - s)_t^E \mapsto (7p + s)_t^D \mapsto (8p + s)_b^L \mapsto \dots
\end{aligned}$$

Each block is determined by a certain range of horizontal positions. For instance, block  $A$  corresponds to positions in the range  $1, \dots, q$ , and similarly block  $I$  has range  $2q + 1, \dots, 3q$ . We claim that the system of inequalities given by the position sequence above admits no solution. Again, a script in Python was our key to quickly understand which inequalities are relevant to establish this fact, and this can be confirmed bare hand or, more quickly, by using more software, like Mathematica.

We give here an explicit way showing the system:

$$\begin{aligned} q &\in \mathbb{N} \setminus \{0\}, \quad p \in \{1, \dots, q\}, \quad s \in \{1, \dots, q\} \\ 3q + 1 &\leq 5p + s \leq 4q \\ 3q + 1 &\leq 8p + s \leq 4q \end{aligned}$$

(which comes from  $(5p + s)_t^D$  and  $(8p + s)_b^L$  on the last but one and last row from the above chain of position, respectively) has no solution. By dividing the last two lines by  $q$ , we get:

$$\begin{aligned} 3 &< 3 + \frac{1}{q} \leq \frac{5p + s}{q} \leq 4 \\ 3 &< 3 + \frac{1}{q} \leq \frac{8p + s}{q} \leq 4 \end{aligned}$$

So both  $\frac{5p+s}{q}$  and  $\frac{8p+s}{q}$  must lie in the interval  $(3, 4]$ . Consequently:

$$\frac{8p + s}{q} - \frac{5p + s}{q} < 1$$

hence

$$3p < q, \text{ thus } p < q/3.$$

We now use this together with

$$5p + s \geq 3q + 1$$

to get

$$\frac{5q}{3} + s > 5p + s \geq 3q + 1.$$

Therefore:

$$s > \frac{4}{3}q + 1 > q,$$

giving a contradiction. This concludes the proof of Proposition 6.2 for all words which contain the pattern (6.2).

For words whose length in  $a$  and  $b$  is greater than five, the argument above applies. If a word has length smaller than 6, though, it does not necessarily contain pattern (6.2); writing down all the words with denominator from 1 to 5, one notices that all of them are blocking, except for the ones corresponding to the rational numbers  $0/1$  and  $1/2$ . When the denominator is greater than 5, the length of the

corresponding words must be at least 6, and so the argument above applies.

Here are the first lifted curve words when  $q \leq 2$ :

$$\begin{aligned} p/q = 0/1, & \quad \text{words: } a, b, c \\ p/q = 1/1, & \quad \text{word: } acb \\ p/q = 1/2, & \quad \text{words: } ac^{-1}, ba^{-1}, cb^{-1} \end{aligned}$$

The pleating rays corresponding to  $0/1$  and  $1/1$  are, respectively, the horizontal semi-line  $(-\infty, -3]$  and the horizontal semi-line  $[2, \infty)$ . Note that we have already discussed these semilines in Section 5.2. The pleating ray corresponding to  $1/2$  is formed of 2 connected components, namely the vertical lines shown in the picture of the slice. The words  $a$ ,  $b$  and  $c$  are clearly primitive.  $acb$  is the commutator word, and obviously primitive blocking. The three words of  $\gamma_{1/2}$  are also primitive.

It is now time to state an important Theorem of Minsky, slightly adapted to our context:

**Theorem 6.8** ([Min], Theorem 4.1). *Let  $\rho$  be a discrete, faithful and geometrically finite representation, with a cusp which is a blocking curve. Then  $\rho$  is primitive stable.*

**Remark 6.9.** *Theorem 6.8 can be tweaked to show that the conclusion is still valid when having any finite number of blocking cusps. For, Minsky shows that simple geodesics associated to primitive elements have to lie within a fixed compact set contained in the compact core of the quotient manifolds. The argument he uses is by contradiction: if one of these geodesics went up the cusp, the chunk of the primitive word which describes this geodesic segment would contain some high power of the cusp blocking word, negating the primitive hypotheses. If instead of one blocking cusp we have many, adapting his argument we easily see that if the primitive geodesic were to enter one of the blocking cusps, then it would contain a high power of the word corresponding to the blocking cusp, again contradicting the primitive property. This is based on the easy observation that if a reduced word contains a subword which is primitive blocking, then the whole word must be primitive blocking too. This is very important to us, since we know that  $\Omega$ -symmetric curves can have three connected components. In this case, as we move along the pleating ray, we pinch simultaneously the three components to get in the boundary of  $\mathcal{DF}$  a quotient with three cusps.*

Theorem 6.3 follows by Proposition 6.2 and Remark 6.9 above.

### 6.3 Irrational case: doubly incompressible laminations are blocking

We have now understood what happens when we fix a simple closed (multi)curve on the surface of the handlebody and pinch it to a parabolic element. By general theory of deformation of Kleinian groups, we know that points in  $\partial\mathcal{DF} \cap \Delta$  which do not correspond to this case, are associated to geometrically infinite representations, which can be described in terms of ending laminations. The reason why this is true is rather deep and complicated, and it is a consequence of the celebrated *tameness theorem* for hyperbolic manifolds. The literature on the topic is vast and fragmented. One recent paper that, among other things, also tries to put together results obtained over the last years is [CG]. In particular, we report here the statement of one of their theorems:

**Theorem 6.10** ([CG], Theorem 0.3). *Let  $N$  be a complete hyperbolic manifold with finitely generated fundamental group. Then every end of  $N$  is geometrically tame, i.e. it is either geometrically finite or simply degenerate.*

In the previous section about the rational case, we have dealt with geometrically finite ends. It is now time to understand what happens for simply degenerate ends. Recall that an end of a manifold is said *simply degenerate* if there exists a sequence of closed geodesics exiting the end. These geodesics converge to a limit lamination, that is called *ending lamination*. Although we shall not need it in what follows, it is worth pointing the reader to another remarkable result in [CG]: the classification theorem for hyperbolic 3-manifolds with finitely generated fundamental group. The proof of the theorem has been carried on by a huge number of people in the last decades. This is the statement in its ultimate form, finally completely proved by Calegari and Gabai.

**Theorem 6.11** ([CG], Classification theorem). *If  $N$  is a complete hyperbolic 3-manifold with finitely generated fundamental group, then  $N$  is determined, up to isometry, by its topological type, the conformal boundary of its geometrically finite ends, and the ending laminations of its geometrically infinite ends*

In order to complete the analysis of the boundary of  $\mathcal{DF} \cap \Delta$ , we need to understand whether characters with simply degenerate ends are primitive stable.

The reference paper for this section will be [JK]. Certain results in this paper are slightly confusing or not completely correct as stated. We shall clarify the meaning of the statements that we need and adjust some small details that

concern our purposes. Let us begin by stating the result we need. The following is a rephrasing of Theorem 4.1 in [JK].

**Theorem 6.12** ([JK], Theorem 4.1). *Let  $\rho: F_n \rightarrow \mathbb{P}SL(2, \mathbb{C})$  be a geometrically infinite representation of the free group of rank  $n$ . If  $\rho(F_n)$  contains no parabolics, then  $\rho$  is primitive-stable.*

Before moving on to discuss the details, we immediately notice that, by applying this theorem to the geometrically infinite representations in  $\partial\mathcal{DF}$ , and combining it with our previous result, Theorem 6.3, we get the final result of this thesis:

**Theorem 6.13.** *Every point in  $\partial_\Delta\mathcal{DF}$ , with the exceptions of the endpoint  $p_1$  of the pleating ray of  $\gamma_{0/1}$  and the two endpoints  $p_2$  and  $p_3$  of the two components of the pleating ray of  $\gamma_{1/2}$ , corresponds to a primitive stable character. In particular,  $\mathcal{PS}$  being an open set and  $\mathcal{DF}$  a closed one, there exists a whole open neighbourhood of  $\mathcal{DF} \setminus \{p_1, p_2, p_3\}$  which is contained in  $\mathcal{PS}$ .*

We now go back to clarifying a couple of points about the proof of Theorem 4.1 from [JK]. At the end of their argument ([JK], p.7), the authors use a lemma previously proved ([JK], Lemma 2.1, p.3). The statement of this lemma says that if  $\lambda$  is a doubly incompressible lamination, then there exists a system of disks  $\Delta$  such that the Whitehead graph  $Wh(\lambda, \Delta)$  is connected and has no cut-points. The first claim of the proof says that the minimum

$$\min_{\Delta} i(\Delta, \lambda)$$

of the intersection number between the fixed lamination  $\lambda$  and  $\Delta$ , as  $\Delta$  varies over all the possible disk systems, is always attained and it is positive. If this were not true, the real sequence of intersection numbers would have to converge to zero, contradicting the assumption of double incompressibility. Although we believe the statement of the lemma to be true, it is not clear whether this specific claim, not motivated in the paper, is true or not. For, at least at a first analysis, it is not clear why such a sequence could not just converge to a positive limit. Nevertheless, there is a simple workaround to make sure that everything works, at least for the strict purposes that we need. In his Ph.D. thesis, Otal proved that if  $\lambda$  is a lamination in the Masur domain, then a disk system that minimises the intersection number with the lamination exists, and that the associated Whitehead graph is connected and with no cut-points. A proof of this fact can also be found in [Min] (Lemma 4.5). It turns out that irrational ending laminations always belong to the Masur domain

([Can], Corollary 10.2). By applying Otal's result, we can therefore conclude that its Whitehead graph, with respect to an appropriate disk system, has the desired connectedness property. Since this is the only crucial property that we need in order to conclude the proof of Theorem 4.1 in [JK], this little tweak is enough to guarantee the validity of the result. Note that a doubly incompressible lamination must belong to the Masur domain, but the former set is strictly larger than the latter. Hence, Jeon and Kim's result, if correct, could be actually considered as an extension of Otal's.

# Bibliography

- [Bow] Bowditch B., *Markoff triples and Quasi-Fuchsian groups*, Proc. London Math. Soc. (3) 77 (1998), no.3, 697-736.
- [CG] Calegari D., Gabai D. *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 19 (2006), no.2, 385-446.
- [Can] Canary R., *Ends of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 6 (1993), no.1, 1-35.
- [CMZ] Cohen M., Metzler W., Zimmermann A. *What does a basis of  $F(a, b)$  look like?*, Math. Ann. 257 (1981), no.4, 435-445.
- [Flo] Floyd W., *Group completions and limit sets of Kleinian groups*, Inventiones Math. 57 (1980), 205-218.
- [Gil] Gilman J., *Two-generator discrete subgroups of  $PSL(2, R)$* , Memoirs of the American Mathematical Society 117 (1995) no.561
- [Gol1] Goldman W., *Ergodic theory on moduli spaces*, Ann. of Math. (2) 146 (1997), no.3, 475-507.
- [Gol2] Goldman W., *The modular group action on real  $SL(2)$ -characters of a one-holed torus*, Geom. Topol. 7 (2003), 443-486.
- [JK] Jeon W., Kim I. *Primitive stable representations of geometrically infinite handlebody hyperbolic 3-manifolds*, arXiv:1003.2055v3 [math.GT] (2011).
- [KS1] Keen L., Series C., *Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori*, Topology 32 (1993), no.4, 719-749.
- [KS2] Keen L., Series C., *Pleating coordinates for the Teichmüller space of a punctured torus*, Bull. Amer. Math. Soc. 26 (1992), no.1, 141-146.



- [KS3] Keen L., Series C., *Pleating invariants for punctured torus groups*, Topology 43 (2004), no.2, 447-491.
- [KS4] Keen L., Series C., *The Riley slice of Schottky space*, Proc. London Math.Soc. (3) 69 (1994), no.1, 72-90.
- [Mat] Mathews D.V., *From Algebra to Geometry: A Hyperbolic Odyssey*, M.Sc. Thesis (2005), [http://www.danielmathews.info/articles/masters\\_thesis.pdf](http://www.danielmathews.info/articles/masters_thesis.pdf).
- [MT] Matsuzaki K. and Taniguchi M., *Hyperbolic manifolds and Kleinian groups*, Oxford University Press (2002).
- [Min] Minsky Y., *On dynamics of  $\text{Out}(F_n)$  on  $\text{PSL}(2, \mathbb{C})$  characters*, Israel J. Math. 193 (2013) no.1, 47-70.
- [MS] Mj M. and Series C., *Limits of limit sets I*, Geom. Dedicata 167 (2013), 35-67.
- [MSW] Mumford D., Series C. and Wright D. *Indra's pearls: the vision of Felix Klein*, Cambridge University Press (2002).
- [Ser] Series C. *The geometry of Markoff numbers*, Math. Intelligencer 7 (3), (1985), 20-29.
- [STY] Series C., Tan S.P. and Yamashita Y. *The diagonal slice of Schottky space*, arXiv:1409.6863v1 [math.GT] (2014).
- [TWZ] Tan S.P., Wong Y.L. and Zhang Y. *Generalized Markoff maps and McShane's identity*, Adv. Math. 217 (2008), no.2, 761-813.